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KIRCHHOFF-TYPE SYSTEM INVOLVING THE $(\alpha_1(m), \alpha_2(m))$-KIRCHHOFF–LAPLACIAN OPERATOR ON THE VARIABLE EXPONENT SOBOLEV SPACES

Abstract. In this paper, we are interested in the existence of weak solutions for a class of Kirchhoff-type systems driven by the $(\alpha_1(m),\alpha_2(m))$-Kirchhoff–Laplacian operator with the Dirichlet boundary conditions as follows:

$$\begin{cases} -\mathcal{R}_1\bigg(\displaystyle\int\limits_{\mathcal{D}} \mathcal{L}_{\psi}^{\alpha_1}\,dm\bigg)\Big(\Delta_{\alpha_1(m)}\psi-|\psi|^{\alpha_1(m)-2}\psi\Big)+\delta_1|\psi|^{p(m)-2}\psi=\lambda_1 f(m,\psi,\nabla\psi) & \text{in } \mathcal{D},\\ -\mathcal{R}_2\bigg(\displaystyle\int\limits_{\mathcal{D}} \mathcal{L}_{\varphi}^{\alpha_2}\,dm\bigg)\Big(\Delta_{\alpha_2(m)}\varphi-|\varphi|^{\alpha_2(m)-2}\varphi\Big)+\delta_2|\varphi|^{q(m)-2}\varphi=\lambda_2 g(m,\varphi,\nabla\varphi) & \text{in } \mathcal{D},\\ \psi=\varphi=0 & \text{on } \partial\mathcal{D}, \end{cases}$$

where $\mathcal{L}_{\psi}^{\alpha_1}$ and $\mathcal{L}_{\varphi}^{\alpha_2}$ are non-local integro-differential operators, $\mathcal{D}$ is an open bounded subset of $\mathbb{R}^N$ with the Lipshcitz boundary $\partial\mathcal{D}$. Under some suitable assumptions on the functions $\mathcal{R}_1$, $\mathcal{R}_2$, f and g, together with the Berkovits topological degree and the variable exponent Sobolev spaces theory, we discuss the existence of weak solutions for the above problem on the spaces $W_0^{1,\alpha_1(m)}(\mathcal{D})\times W_0^{1,\alpha_2(m)}(\mathcal{D})$.

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რეზიუმე. ნაშრომში შესწავლილია სუსტი ამონახსნების არსებობა კირკჰოფის ტიპის სისტემათა ერთი კლასისთვის, რომელიც დაკავშირებულია $(\alpha_1(m),\alpha_2(m))$-კირკჰოფ-ლაპლასის ოპერატორთან დირიხლეს სასაზღვრო პირობებით:

$$\begin{cases} -\mathcal{R}_1\bigg(\displaystyle\int\limits_{\mathcal{D}} \mathcal{L}_{\psi}^{\alpha_1}\,dm\bigg)\Big(\Delta_{\alpha_1(m)}\psi-|\psi|^{\alpha_1(m)-2}\psi\Big)+\delta_1|\psi|^{p(m)-2}\psi=\lambda_1 f(m,\psi,\nabla\psi) & \mathcal{D}\text{-ში},\\ -\mathcal{R}_2\bigg(\displaystyle\int\limits_{\mathcal{D}} \mathcal{L}_{\varphi}^{\alpha_2}\,dm\bigg)\Big(\Delta_{\alpha_2(m)}\varphi-|\varphi|^{\alpha_2(m)-2}\varphi\Big)+\delta_2|\varphi|^{q(m)-2}\varphi=\lambda_2 g(m,\varphi,\nabla\varphi) & \mathcal{D}\text{-ში},\\ \psi=\varphi=0 & \mathcal{D}\text{-ზე}, \end{cases}$$

სადაც $\mathcal{L}_{\psi}^{\alpha_1}$ და $\mathcal{L}_{\varphi}^{\alpha_2}$ არალოკალური ინტეგრო-დიფერენციალური ოპერატორებია, ხოლო $\mathcal{D}$ არის $\mathbb{R}^N$-ის ღია შემოსაზღვრული ქვესიმრავლე ლიფშიცის საზღვრით $\partial\mathcal{D}$. $\mathcal{R}_1$, $\mathcal{R}_2$, f და g ფუნქციებზე რამდენიმე შესაფერისი დაშვების პირობებში, ბერკოვიცის ტოპოლოგიური ხარისხისა და სობოლევის ცვლად მაჩვენებლიან სივრცეთა თეორიის გამოყენებით, განხილულია ზემოაღნიშნული ამოცანის სუსტი ამონახსნების არსებობა $W_0^{1,\alpha_1(m)}(\mathcal{D})\times W_0^{1,\alpha_2(m)}(\mathcal{D})$ სივრცეებზე.

1 Introduction

The study of differential equations and variational problems with a variable exponent is a very recent and exciting field. It was initiated by the theory of nonlinear elasticity, stationary thermorheological viscous flows, electrorheological fluids and image restorations. These areas have received increasing attention and lot of results have been published by many authors (see [1,4,32,35–37,40]). In particular, the nonlocal elliptic Kirchhoff type problems [22,27,39] were first considered by Kirchhoff in 1883 [25]. More specifically, Kirchhoff has suggested a model represented by

$$\rho\psi_{tt} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\psi_m|^2\, dm\right)\psi_{mm} = 0.$$

Expanding upon the classical d'Alembert's wave equation, this extension takes into account the variations in the string's length as it undergoes vibrations.

Our principal objective in this paper is to study the problems with a variable exponent and a non-homogeneous differential operator in the sense that our operators include not only the case of the α-Laplacian operator or the (α_1, α_2)-Laplacian operator considered by several authors [5, 6, 12–14, 28, 29, 31, 34], but also other types of operators. Based on the cited papers, many interesting questions arise from the search for weak solutions for a non-local elliptic Kirchhoff system involving the $(\alpha_1(m), \alpha_2(m))$-Laplacian operators in bounded domains with Dirichlet boundary conditions.

In details, we consider the following problem:

$$\begin{cases} -\mathcal{R}_1\left(\int\limits_{\mathcal{D}} \mathcal{L}_\psi^{\alpha_1}\, dm\right)\left(\Delta_{\alpha_1(m)}\psi - |\psi|^{\alpha_1(m)-2}\psi\right) + \delta_1|\psi|^{p(m)-2}\psi = \lambda_1 f(m, \psi, \nabla\psi) & \text{in } \mathcal{D}, \\ -\mathcal{R}_2\left(\int\limits_{\mathcal{D}} \mathcal{L}_\varphi^{\alpha_2}\, dm\right)\left(\Delta_{\alpha_2(m)}\varphi - |\varphi|^{\alpha_2(m)-2}\varphi\right) + \delta_2|\varphi|^{q(m)-2}\varphi = \lambda_2 g(m, \varphi, \nabla\varphi) & \text{in } \mathcal{D}, \\ \psi = \varphi = 0 & \text{on } \partial\mathcal{D}, \end{cases} \tag{1.1}$$

where $\mathcal{L}_\psi^{\alpha_1}$ and $\mathcal{L}_\varphi^{\alpha_2}$ are given by

$$\mathcal{L}_\psi^{\alpha_1} := \frac{|\nabla\psi|^{\alpha_1(m)} + |\psi|^{\alpha_1(m)}}{\alpha_1(m)} \quad \text{and} \quad \mathcal{L}_\varphi^{\alpha_1} := \frac{|\nabla\varphi|^{\alpha_2(m)} + |\varphi|^{\alpha_2(m)}}{\alpha_2(m)},$$

$\mathcal{D}$ is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 2)$ and λ_i, δ_i $(i = 1, 2)$ are the real parameters, $\mathcal{R}_1$ and $\mathcal{R}_2$ are the functions satisfying the assumptions that will be given later, and the second term is divided into two convection terms.

The $\alpha(m)$-Laplacian operator $\Delta_{\alpha(m)}\psi := \operatorname{div}(|\nabla\psi|^{\alpha(m)-2}\nabla\psi)$ possesses more complex nonlinear behavior compared to the p-Laplacian operator, primarily because it lacks homogeneity, this implies that the lowest eigenvalues of the $\alpha(m)$-Laplacian are not zero (see also [33]). Motivated by [1, 6, 15–18, 22, 29, 30], our primary objective is to prove the existence of weak solutions for a nonlocal elliptic system involving the $(\alpha_1(m), \alpha_2(m))$-Kirchhoff–Laplacian operators depending on four real parameters under the Dirichlet boundary condition, by employing another approach that relies on the Berkovits topological degree in the framework of Sobolev space with a variable exponent.

The structure of this paper is outlined as follows. In Section 2, we provide a brief overview of fundamental concepts in the functional framework and introduce various types of generalized (S_+) operators, together with the Berkovits topological degree. In Section 3, we give the result of this paper and its proof.

2 Preliminaries

2.1 Variable exponent Sobolev spaces

In this subsection, we discuss several fundamental properties of variable exponent Sobolev spaces. For more background information on this subject, we refer to [8, 9, 19–21, 26].

Define the generalized Lebesgue space by

$$L^{\alpha(m)}(\mathcal{D}) = \Big\{\varphi : \ \mathcal{D} \to \mathbb{R} \ \text{ measurable function}, \ \int\limits_{\mathcal{D}} |\varphi(m)|^{\alpha(m)}\, dm < +\infty\Big\},$$

where $\alpha \in \mathrm{C}_+(\overline{\mathcal{D}})$ and

$$\mathrm{C}_+(\overline{\mathcal{D}}) = \big\{\alpha : \ \alpha \in C(\overline{\mathcal{D}}), \ \alpha(m) > 1 \ \text{ for every } \ m \in \overline{\mathcal{D}}\big\}.$$

For any $\alpha \in \mathrm{C}_+(\overline{\mathcal{D}})$, we establish

$$\alpha^+ := \max\big\{\alpha(m), m \in \overline{\mathcal{D}}\big\}, \quad \alpha^- := \min\big\{\alpha(m), m \in \overline{\mathcal{D}}\big\}.$$

The space $L^{\alpha(m)}(\mathcal{D})$ is equipped by the norm

$$|\varphi|_{\alpha(m)} = \inf\Big\{k > 0 : \ \rho_{\alpha(m)}\Big(\frac{\varphi}{k}\Big) \leq 1\Big\},$$

where the modular $\rho_{\alpha(m)} : L^{\alpha(m)}(\mathcal{D}) \to \mathbb{R}$ is defined by

$$\rho_{\alpha(m)}(\varphi) = \int\limits_{\mathcal{D}} |\varphi(m)|^{\alpha(m)}\, dm, \ \ \forall\, \varphi \in L^{\alpha(m)}(\mathcal{D}),$$

and satisfies some useful properties listed below.

Proposition 2.1 ([21])**.** *Let (φ_k) and $\varphi \in L^{\alpha(m)}(\mathcal{D})$, then*

$$|\varphi|_{\alpha(m)} < 1 \ \ (\textit{resp.} = 1; \ > 1) \Longleftrightarrow \rho_{\alpha(m)}(\varphi) < 1 \ \ (\textit{resp.} = 1; \ > 1), \tag{2.1}$$

$$|\varphi|_{\alpha(m)} > 1 \implies |\varphi|^{\alpha^-}_{\alpha(m)} \leq \rho_{\alpha(m)}(\varphi) \leq |\varphi|^{\alpha^+}_{\alpha(m)}, \tag{2.2}$$

$$|\varphi|_{\alpha(m)} < 1 \implies |\varphi|^{\alpha^+}_{\alpha(m)} \leq \rho_{\alpha(m)}(\varphi) \leq |\varphi|^{\alpha^-}_{\alpha(m)}, \tag{2.3}$$

$$\lim_{k\to\infty} |\varphi_k - \varphi|_{\alpha(m)} = 0 \Longleftrightarrow \lim_{k\to\infty} \rho_{\alpha(m)}(\varphi_k - \varphi) = 0. \tag{2.4}$$

Remark 2.1. Take note that, based on equations (2.2) and (2.3), we can deduce the inequalities

$$|\varphi|_{\alpha(m)} \leq \rho_{\alpha(m)}(\varphi) + 1, \tag{2.5}$$

$$\rho_{\alpha(m)}(\varphi) \leq |\varphi|^{\alpha^-}_{\alpha(m)} + |\varphi|^{\alpha^+}_{\alpha(m)}. \tag{2.6}$$

Proposition 2.2 ([11,26])**.** *$L^{\alpha(m)}(\mathcal{D})$ is a separable and reflexive Banach space.*

Proposition 2.3 ([26])**.** *Define $L^{\alpha'(m)}(\mathcal{D})$ as the space conjugate of $L^{\alpha(m)}(\mathcal{D})$, where $\frac{1}{\alpha(m)} + \frac{1}{\alpha'(m)} = 1$ for all $m \in \mathcal{D}$. For any $\psi \in L^{\alpha(m)}(\mathcal{D})$ and $\varphi \in L^{\alpha'(m)}(\mathcal{D})$, we have the Hölder inequality*

$$\Big|\int\limits_{\mathcal{D}} \psi\varphi\, dm\Big| \leq \Big(\frac{1}{\alpha-} + \frac{1}{\alpha'-}\Big)|\psi|_{\alpha(m)}|\varphi|_{\alpha'(m)} \leq 2|\psi|_{\alpha(m)}|\varphi|_{\alpha'(m)}. \tag{2.7}$$

Remark 2.2. If $\alpha_1, \alpha_2 \in \mathrm{C}_+(\overline{\mathcal{D}})$ with $\alpha_1(m) \leq \alpha_2(m)$ for any $m \in \overline{\mathcal{D}}$, then there exists a continuous embedding $L^{\alpha_2(m)}(\mathcal{D}) \hookrightarrow L^{\alpha_1(m)}(\mathcal{D})$.

Now, we define the space $W^{1,\alpha(m)}(\mathcal{D})$ by

$$W^{1,\alpha(m)}(\mathcal{D}) = \big\{\varphi \in L^{\alpha(m)}(\mathcal{D}) : \ |\nabla\varphi| \in L^{\alpha(m)}(\mathcal{D})\big\},$$

equipped with the norm

$$\|\varphi\|_{\alpha(m)} = |\varphi|_{\alpha(m)} + |\nabla\varphi|_{\alpha(m)}.$$

We denote by $W_0^{1,\alpha(m)}(\mathcal{D})$ the closure of $C_0^\infty(\mathcal{D})$ with respect to the norm of $W^{1,\alpha(m)}(\mathcal{D})$.

Proposition 2.4 ([23, 38]). *If the exponent $\alpha(m)$ satisfies the log-Hölder continuity condition, i.e., there exists $\tau > 0$ such that for each $m_1, m_2 \in \mathcal{D}$, $m_1 \neq m_2$ with $|m_1 - m_2| \leq \frac{1}{2}$, we have*

$$|\alpha(m_1) - \alpha(m_2)| \leq \frac{\tau}{-\log|m_1 - m_2|}, \tag{2.8}$$

then there exists a constant $C_{\mathcal{G},\alpha} > 0$ such that

$$|\varphi|_{\alpha(m)} \leq C_{\mathcal{G},\alpha} |\nabla\varphi|_{\alpha(m)}, \quad \forall\, \varphi \in W_0^{1,\alpha(m)}(\mathcal{D}). \tag{2.9}$$

Therefore, we could use on $W_0^{1,\alpha(m)}(\mathcal{D})$ the following norm

$$|\varphi|_{1,\alpha(m)} = |\nabla\varphi|_{\alpha(m)},$$

which is equivalent to $\|\varphi\|_{\alpha(m)}$.

Proposition 2.5 ([10, 26]). *The spaces $W^{1,\alpha(m)}(\mathcal{D})$ and $W_0^{1,\alpha(m)}(\mathcal{D})$ are separable and reflexive Banach spaces.*

Remark 2.3. The dual space of $W_0^{1,\alpha(m)}(\mathcal{D})$ denoted by $W^{-1,\alpha'(m)}(\mathcal{D})$ is equipped with the norm

$$|\varphi|_{-1,\alpha'(m)} = \inf\Big\{|\varphi_0|_{\alpha'(m)} + \sum_{j=1}^{N} |\varphi_j|_{\alpha'(m)}\Big\},$$

where the infnimum is taken on all possible decompositions $\varphi = \varphi_0 - \operatorname{div}\mathcal{L}$ with $\varphi_0 \in L^{\alpha'(m)}(\mathcal{D})$ and $\mathcal{L} = (\varphi_1, \dots, \varphi_N) \in (L^{\alpha'(m)}(\mathcal{D}))^N$.

In the subsequent discussions, we will make use of the Cartesian product space

$$\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)} := W_0^{1,\alpha_1(m)}(\mathcal{D}) \times W_0^{1,\alpha_2(m)}(\mathcal{D}),$$

which is equipped with the norm

$$\|(\psi, \varphi)\|_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}} = \max\big\{\|\psi\|_{\alpha_1(m)}, \|\varphi\|_{\alpha_2(m)}\big\},$$

where $\|\cdot\|_{\alpha_1(m)}$ is the norm of $W_0^{1,\alpha_1(m)}(\mathcal{D})$ and $\|\cdot\|_{\alpha_2(m)}$ is the norm of $W_0^{1,\alpha_2(m)}(\mathcal{D})$.

The space $(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)})^*$ is the dual space of $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$ corresponding to the Orlicz–Sobolev space $W_0^{-1,\alpha_1'(m)}(\mathcal{D}) \times W_0^{-1,\alpha_2'(m)}(\mathcal{D})$ equipped with the norm

$$\|\cdot\|_{(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)})^*} := \max\big\{\|\cdot\|_{-1,\alpha_1'(m)}, \|\cdot\|_{-1,\alpha_2'(m)}\big\}.$$

The continuous pairing between the space $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$ and its dual space $(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)})^*$ is given by

$$\langle\cdot,\cdot\rangle_{\alpha_1,\alpha_2} = \langle\cdot,\cdot\rangle_{1,\alpha_1(m)} + \langle\cdot,\cdot\rangle_{1,\alpha_2(m)}.$$

2.2 Topological degree theory

Now, our aim is to examine several definitions and fundamental properties of Berkovits degree theory applied to demicontinuous operators in a reflexive space over the real numbers.

Let $\mathcal{G}$ be a real separable reflexive Banach space and $\mathcal{E}$ be a nonempty subset of $\mathcal{G}$. The symbol $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ means the usual dual paring between $\mathcal{G}^*$ and $\mathcal{G}$.

Definition 2.1. Let E be a second real Banach space. A mapping $\mathcal{B} : \mathcal{E} \subset \mathcal{G} \to E$ is

- bounded if it transforms any bounded set into a bounded set.
- demicontinuous if for any $(\varphi_k) \subset \mathcal{E}$, $\varphi_k \to \varphi$ then $\mathcal{B}(\varphi_k) \rightharpoonup \mathcal{B}(\varphi)$.

- compact if $\mathcal{B}$ is continuous and for any $A \subset \mathcal{G}$ bounded we have $\mathcal{B}(A)$ is relatively compact.

Definition 2.2. An operator $\mathcal{B} : \mathcal{E} \subset \mathcal{G} \to \mathcal{G}^*$ is called

- of type (S_+) if for any $(\varphi_k) \subset \mathcal{E}$ with $\varphi_k \rightharpoonup \varphi$ and $\limsup_{k\to\infty} \langle \mathcal{B}(\varphi_k), \varphi_k - \varphi\rangle \leq 0$, we have $\varphi_k \to \varphi$.
- quasimonotone if for any sequence $(\varphi_k) \subset \mathcal{E}$ with $\varphi_k \rightharpoonup \varphi$, we have $\limsup_{k\to\infty} \langle \mathcal{B}(\varphi_k), \varphi_k - \varphi\rangle \geq 0$.

Definition 2.3. Let $\mathrm{Z} : \mathcal{E}_1 \subset \mathcal{G} \to \mathcal{G}^*$ be a bounded mapping with $\mathcal{E} \subset \mathcal{E}_1$. For any operator $\mathcal{B} : \mathcal{E} \subset \mathcal{G} \to \mathcal{G}$, we can say that

- $\mathcal{B}$ satisfies condition $(S_+)_{\mathrm{Z}}$ if for any $(\varphi_k) \subset \mathcal{E}$ with $\varphi_k \rightharpoonup \varphi$, $\psi_k := \mathrm{Z}(\varphi_k) \rightharpoonup \psi$ and $\limsup_{k\to\infty} \langle \mathcal{B}(\varphi_k), \psi_k - \psi\rangle \leq 0$, we have $\varphi_k \to \varphi$.
- $\mathcal{B}$ possess the property $(QM)_{\mathrm{Z}}$ if for any sequence $(\varphi_k) \subset \mathcal{E}$ with $\varphi_k \rightharpoonup \varphi$, $\psi_k := \mathrm{Z}(\varphi_k) \rightharpoonup \psi$, we have $\limsup_{k\to\infty} \langle \mathcal{B}(\varphi_k), \psi - \psi_k\rangle \geq 0$.

Next, we consider the following sets of operators:

$$
\begin{aligned}
\mathcal{T}_1(\mathcal{E}) &:= \big\{\mathcal{B} : \mathcal{E} \to \mathcal{G}^* : \ \mathcal{B} \text{ is of type } (S_+) \text{ and is a demicontinuous and bounded map}\big\},\\
\mathcal{T}_{\mathrm{Z}}(\mathcal{E}) &:= \big\{\mathcal{B} : \mathcal{E} \to \mathcal{G} : \ \mathcal{B} \text{ satisfies the condition } (S_+)_{\mathrm{Z}} \text{ and is a demicontinuous map}\big\},\\
\mathcal{T}_{\mathrm{Z},B}(\mathcal{E}) &:= \big\{\mathcal{B} \in \mathcal{T}_{\mathrm{Z}}(\mathcal{E}) \text{ such that } \mathcal{B} \text{ is a bounded map}\big\},
\end{aligned}
$$

for any $\mathcal{E} \subset D(\mathcal{B})$ and any $\mathrm{Z} \in \mathcal{T}_1(\mathcal{E})$, where $D(\mathcal{B})$ is the domain of $\mathcal{B}$.

Set

$$
\mathcal{T}(\mathcal{G}) := \big\{\mathcal{B} \in \mathcal{T}_{\mathrm{Z}}(\overline{E}) : \ E \in \mathcal{O}, \ \mathrm{Z} \in \mathcal{T}_1(\overline{E})\big\},
$$

where $\mathrm{Z} \in \mathcal{T}_1(\overline{\mathrm{E}})$ is known as an essential inner map to $\mathcal{B}$ and $\mathcal{O}$ is the collection of all bounded open sets in $\mathcal{G}$.

Lemma 2.1 ([24])**.** *Let $\mathrm{Z} \in \mathcal{T}_1(\overline{E})$ be continuous and $\mathcal{P} : D(\mathcal{P}) \subset \mathcal{G}^* \to \mathcal{G}$ be demicontinuous such that $\mathrm{Z}(\overline{E}) \subset D(\mathcal{P})$, where E is a bounded open set in a real reflexive Banach space $\mathcal{G}$. Therefore, the assertions below are correct:*

- *If $\mathcal{P}$ is quasimonotone, then $I + \mathcal{P} \circ \mathrm{Z} \in \mathcal{T}_Z(\overline{E})$, where the identity operator is denoted by I.*
- *If $\mathcal{P}$ is of type (S_+), then $\mathcal{P} \circ \mathrm{Z} \in \mathcal{T}_Z(\overline{E})$.*

Definition 2.4. Assume that E is a bounded open subset of a real reflexive Banach space $\mathcal{G}$, $\mathrm{Z} \in \mathcal{T}_1(\overline{E})$ is continuous and let $\mathcal{B}, \mathcal{P} \in \mathcal{T}_Z(\overline{E})$. Then $\mathcal{H} : [0,1] \times \overline{E} \to \mathcal{G}$ defined by

$$
\mathcal{H}(t,\varphi) := (1-t)\mathcal{B}\varphi + t\mathcal{P}\varphi \ \text{ for } \ (t,\varphi) \in [0,1] \times \overline{E}
$$

is called an admissible affine homotopy with the common continuous essential inner map Z.

Remark 2.4 ([24])**.** The affine homotopy $\mathcal{H}$ satisfies the condition $(S_+)_{\mathrm{Z}}$.

Next, as in [24], we introduce the topological degree in the class $\mathcal{T}(\mathcal{G})$.

Theorem 2.1. *Let*

$$
D = \big\{(\mathcal{P}, E, h) : \ E \in \mathcal{O}, \ \mathrm{Z} \in \mathcal{T}_1(\overline{E}), \ \mathcal{P} \in \mathcal{T}_{Z,B}(\overline{E}), \ h \notin \mathcal{P}(\partial E)\big\}.
$$

Then there exists a unique degree function $d : D \to \mathbb{Z}$ that satisfies the following properties:

(1) (Normalization) *For any $h \in \mathcal{P}(E)$, we find that*

$$
d(I, E, h) = 1.
$$

(2) (Homotopy invariance) *If* $\mathcal{H} : [0,1] \times \overline{E} \to \mathcal{G}$ *is a bounded admissible affine homotopy with a common continuous essential inner map and* $h : [0,1] \to \mathcal{G}$ *is a continuous path in* $\mathcal{G}$ *such that* $h(t) \notin \mathcal{H}(t, \partial E)$ *for all* $t \in [0,1]$*, then*

$$d\big(\mathcal{H}(t, \cdot), E, h(t)\big) = C \ \text{ for all } \ t \in [0,1].$$

(3) (Existence) *If* $d(\mathcal{P}, E, h) \neq 0$*, then the equation* $\mathcal{P}\varphi = h$ *has a solution in* E*.*

Definition 2.5 ([24])**.** The above degree is defined as follows:

$$d(\mathcal{P}, E, h) := d_B\big(\mathcal{P}|_{\overline{E}_0}, E_0, h\big),$$

where d_B is the Berkovits degree [2] *and* E_0 *is any open subset of* E *with* $\mathcal{P}^{-1}(h) \subset E_0$ *and* $\mathcal{P}$ *is bounded on* $\overline{E}_0$.

3 Hypotheses and main results

This section focuses on the existence of a weak solution to (1.1). In order to achieve this, we present the assumptions linked to our problem. Assumed $\mathcal{D} \subset \mathbb{R}^N (N \geq 2)$ to be a bounded smooth domain and $\alpha_1, \alpha_2 \in C_+(\overline{\mathcal{D}})$ satisfying (2.8), $p, q \in C_+(\overline{\mathcal{D}})$ with $2 \leq p^- \leq p(m) \leq p^+ < \alpha_1^-$ and $2 \leq q^- \leq q(m) \leq q^+ < \alpha_2^-$, then we impose that

(A_1) $f,\, g : \mathcal{D} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are two Carathéodory functions.

(A_2) There exist $\beta_1 > 0$ and $\gamma_1 \in L^{\alpha_1(m)}(\mathcal{D})$ such that

$$|f(m, \eta, \omega)| \leq \beta_1\big(\gamma_1(m) + |\eta|^{r(m)-1} + |\omega|^{r(m)-1}\big)$$

for a.e. $m \in \mathcal{D}$ and all $(\eta, \omega) \in \mathbb{R} \times \mathbb{R}^N$, where $2 \leq r^- \leq r(m) \leq r^+ < \alpha_1^-$.

(A_3) There exist $\beta_2 > 0$ and $\gamma_2 \in L^{\alpha_2(m)}(\mathcal{D})$ such that

$$|g(m, \eta, \omega)| \leq \beta_2\big(\gamma_2(m) + |\eta|^{s(m)-1} + |\omega|^{s(m)-1}\big)$$

for a.e. $m \in \mathcal{D}$ and all $(\eta, \omega) \in \mathbb{R} \times \mathbb{R}^N$, where $2 \leq s^- \leq s(m) \leq s^+ < \alpha_2^-$.

(M_0) We assume that $\mathcal{R}_i : [0, +\infty) \to [0, +\infty)$ $(i = 1, 2)$ are increasing and continuous functions such that

$$\varepsilon_1 t^{a_1(m)-1} \leq \mathcal{R}_1(t) \leq \varepsilon_2 t^{a_1(m)-1},$$
$$\tau_1 t^{a_2(m)-1} \leq \mathcal{R}_2(t) \leq \tau_2 t^{a_2(m)-1},$$

where ε_i, τ_i $(i = 1, 2)$ are real numbers such that $\varepsilon_1 \leq \varepsilon_2$, $\tau_1 \leq \tau_2$ and $a_1(m)$, $a_2(m) \geq 1$.

In this study, we adopt the following definition for a weak solution of (1.1).

Definition 3.1. We say that $(\psi, \varphi) \in \mathcal{W}_0^{\alpha_1(m), \alpha_2(m)}$ is a weak solution of problem (1.1) if

$$\begin{aligned}
&\mathcal{R}_1\bigg(\int_{\mathcal{D}} \frac{|\nabla\psi|^{\alpha_1(m)} + |\psi|^{\alpha_1(m)}}{\alpha_1(m)}\, dm\bigg) \int_{\mathcal{D}} \Big(|\nabla\psi|^{\alpha_1(m)-2}\nabla\psi\nabla\vartheta + |\psi|^{\alpha_1(m)-2}\psi\vartheta\Big)\, dm \\
&\qquad + \mathcal{R}_2\bigg(\int_{\mathcal{D}} \frac{|\nabla\varphi|^{\alpha_2(m)} + |\varphi|^{\alpha_2(m)}}{\alpha_2(m)}\, dm\bigg) \int_{\mathcal{D}} \Big(|\nabla\varphi|^{\alpha_2(m)-2}\nabla\varphi\nabla\zeta + |\varphi|^{\alpha_2(m)-2}\varphi\zeta\Big)\, dm \\
&\quad = \int_{\mathcal{D}} \Big(-\delta_1|\psi|^{p(m)-2}\psi + \lambda_1 f(m, \psi, \nabla\psi)\Big)\vartheta\, dm + \int_{\mathcal{D}} \Big(-\delta_2|\varphi|^{q(m)-2}\varphi + \lambda_2 g(m, \varphi, \nabla\varphi)\Big)\zeta\, dm
\end{aligned}$$

for each $(\vartheta, \zeta) \in \mathcal{W}_0^{\alpha_1(m), \alpha_2(m)}$.

Next, we introduce two lemmas that will be used in the demonstration of the main outcome.

Lemma 3.1. *If (M_0) holds, then the operator $\mathcal{S}:\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}\to(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)})^*$ defined by*

$$\begin{aligned}
&\big\langle \mathcal{S}(\psi,\varphi),(\vartheta,\zeta)\big\rangle\\
&\quad=\mathcal{R}_1\Bigg(\int\limits_{\mathcal{D}}\frac{|\nabla\psi|^{\alpha_1(m)}+|\psi|^{\alpha_1(m)}}{\alpha_1(m)}\,dm\Bigg)\int\limits_{\mathcal{D}}\Big(|\nabla\psi|^{\alpha_1(m)-2}\nabla\psi\nabla\vartheta+|\psi|^{\alpha_1(m)-2}\psi\vartheta\Big)\,dm\\
&\qquad+\mathcal{R}_2\Bigg(\int\limits_{\mathcal{D}}\frac{|\nabla\varphi|^{\alpha_2(m)}+|\varphi|^{\alpha_2(m)}}{\alpha_2(m)}\,dm\Bigg)\int\limits_{\mathcal{D}}\Big(|\nabla\varphi|^{\alpha_2(m)-2}\nabla\varphi\nabla\zeta+|\varphi|^{\alpha_2(m)-2}\varphi\zeta\Big)\,dm
\end{aligned}$$

is of type (S_+), continuous, bounded, coercive and strictly monotone.

Proof. Let $(\psi,\varphi)\in\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$ and let the functional $\varPsi$ defined on $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$ by

$$\varPsi(\psi,\varphi):=\widehat{\mathcal{R}}_1\Bigg(\int\limits_{\mathcal{D}}\frac{|\nabla\psi|^{\alpha_1(m)}+|\psi|^{\alpha_1(m)}}{\alpha_1(m)}\,dm\Bigg)+\widehat{\mathcal{R}}_2\Bigg(\int\limits_{\mathcal{D}}\frac{|\nabla\varphi|^{\alpha_2(m)}+|\varphi|^{\alpha_2(m)}}{\alpha_2(m)}\,dm\Bigg)$$

where

$$\widehat{\mathcal{R}}_i(s)=\int\limits_0^s\mathcal{R}_i(\tau)\,\mathrm{d}\tau\quad(i=1,2),$$

is continuously Gâteaux differentiable whose Gâteaux derivative at the point $(\psi,\varphi)\in\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$ is the functional $\mathcal{S}:=\varPsi'(\psi,\varphi)\in(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)})^*$ given by

$$\big\langle\mathcal{S}(\psi,\varphi),(\vartheta,\zeta)\big\rangle=\big\langle\mathcal{S}_{\alpha_1}(\psi),(\vartheta)\big\rangle+\big\langle\mathcal{S}_{\alpha_2}(\varphi),(\zeta)\big\rangle,$$

where

$$\big\langle\mathcal{S}_{\alpha_1}(\psi),(\vartheta)\big\rangle=\mathcal{R}_1\Bigg(\int\limits_{\mathcal{D}}\frac{|\nabla\psi|^{\alpha_1(m)}+|\psi|^{\alpha_1(m)}}{\alpha_1(m)}\,dm\Bigg)\int\limits_{\mathcal{D}}\Big(|\nabla\psi|^{\alpha_1(m)-2}\nabla\psi\nabla\vartheta+|\psi|^{\alpha_1(m)-2}\psi\,\vartheta\Big)\,dm$$

and

$$\big\langle\mathcal{S}_{\alpha_2}(\varphi),(\zeta)\big\rangle=\mathcal{R}_2\Bigg(\int\limits_{\mathcal{D}}\frac{|\nabla\varphi|^{\alpha_2(m)}+|\varphi|^{\alpha_2(m)}}{\alpha_2(m)}\,dm\Bigg)\int\limits_{\mathcal{D}}\Big(|\nabla\varphi|^{\alpha_2(m)-2}\nabla\varphi\nabla\zeta+|\varphi|^{\alpha_2(m)-2}\varphi\,\zeta\Big)\,dm.$$

So, $\mathcal{S}$ is bounded, continuous and, since $\mathcal{S}_{\alpha_1}$ and $\mathcal{S}_{\alpha_2}$ are strictly monotone (see [7, Theorem 2.1]), $\mathcal{S}$ is strictly monotone.

Let us prove that $\mathcal{S}$ exhibits coercivity. Utilizing (2.5) and (2.6), we obtain the following expression:

$$\begin{aligned}
\frac{\langle\mathcal{S}(\psi,\varphi),(\psi,\varphi)\rangle}{\|(\psi,\varphi)\|}&=\frac{\mathcal{R}_1\big(\int\limits_{\mathcal{D}}\frac{|\nabla\psi|^{\alpha_1(m)}+|\psi|^{\alpha_1(m)}}{\alpha_1(m)}\big)\int\limits_{\mathcal{D}}(|\nabla\psi|^{\alpha_1(m)}+|\psi|^{\alpha_1(m)})}{\|(\psi,\varphi)\|}\,dm\\
&\quad+\frac{\mathcal{R}_2\big(\int\limits_{\mathcal{D}}\frac{|\nabla\varphi|^{\alpha_2(m)}+|\varphi|^{\alpha_2(m)}}{\alpha_2(m)}\big)\int\limits_{\mathcal{D}}(|\nabla\varphi|^{\alpha_2(m)}+|\varphi|^{\alpha_2(m)})}{\|(\psi,\varphi)\|}\,dm\\
&\geq\frac{\frac{\varepsilon_1}{(\alpha_1^+)^{a_1^--1}}\min\big(\|\psi\|_{\alpha_1(m)}^{\alpha_1^-(a_1^--1)},\|\psi\|_{\alpha_1(m)}^{\alpha_1+(a_1^--1)}\big)\min\big(\|\psi\|_{\alpha_1(m)}^{\alpha_1^-},\|\psi\|_{\alpha_1(m)}^{\alpha_1^+}\big)}{\|(\psi,\varphi)\|}\\
&\quad+\frac{\frac{\tau_1}{(\alpha_2^+)^{a_2^--1}}\min\big(\|\varphi\|_{\alpha_2(m)}^{\alpha_2^-(a_2^--1)},|\varphi\|_{\alpha_2(m)}^{\alpha_2^+(a_2^--1)}\big)\min\big(\|\varphi\|_{\alpha_2(m)}^{\alpha_2^-},\|\varphi\|_{\alpha_2(m)}^{\alpha_2^+}\big)}{\|(\psi,\varphi)\|}.
\end{aligned}$$

This implies that

$$\lim_{\|(\psi,\varphi)\|\to+\infty} \frac{\langle \mathcal{S}(\psi,\varphi),(\psi,\varphi)\rangle}{\|(\psi,\varphi)\|} = +\infty,$$

hence $\mathcal{S}$ is coercive on $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$.

Now, we will demonstrate that the operator $\mathcal{S}$ satisfies the properties of being of type (S_+).

Let $(\psi_k,\varphi_k) \subset \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$ with $(\psi_k,\varphi_k) \rightharpoonup (\psi,\varphi)$ and

$$\overline{\lim_{k\longrightarrow\infty}} \big\langle \mathcal{S}(\psi_k,\varphi_k) - \mathcal{S}(\psi,\varphi), (\psi_k-\psi, \varphi_k-\varphi)\big\rangle \leqslant 0,$$

we demonstrate that $(\psi_k,\varphi_k) \to (\psi,\varphi)$.

If $(\psi_k,\varphi_k) \rightharpoonup (\psi,\varphi)$ and $\overline{\lim_{k\longrightarrow\infty}} \langle \mathcal{S}(\psi_k,\varphi_k) - \mathcal{S}(\psi,\varphi), (\psi_k-\psi, \varphi_k-\varphi)\rangle \leqslant 0$, then

$$\lim_{k\to\infty} \big\langle \mathcal{S}(\psi_k,\varphi_k) - \mathcal{S}(\psi,\varphi), (\psi_k-\psi, \varphi_k-\varphi)\big\rangle = 0.$$

Therefore,

$$\big\langle \mathcal{S}_{\alpha_1}(\psi_k) - \mathcal{S}_{\alpha_1}(\psi), \psi_k-\psi\big\rangle + \big\langle \mathcal{S}_{\alpha_2}(\varphi_k) - \mathcal{S}_{\alpha_2}(\varphi), \varphi_k-\varphi\big\rangle \to 0.$$

Since $\mathcal{S}_{\alpha_1}$ and $\mathcal{S}_{\alpha_2}$ are monotone, we have

$$\big\langle \mathcal{S}_{\alpha_1}(\psi_k) - \mathcal{S}_{\alpha_1}(\psi), \psi_k-\psi\big\rangle \to 0, \quad \big\langle \mathcal{S}_{\alpha_2}(\varphi_k) - \mathcal{S}_{\alpha_2}(\varphi), \varphi_k-\varphi\big\rangle \to 0. \tag{3.1}$$

Following a similar line of reasoning as in [7], we get

$$\begin{aligned}
&\big\langle \mathcal{S}_{\alpha_1}(\psi_k) - \mathcal{S}_{\alpha_1}(\psi), \psi_k-\psi\big\rangle \\
&\geq \mathcal{R}_1\bigg(\int_{\mathcal{D}} \frac{|\nabla\psi|^{\alpha_1(m)} + |\psi|^{\alpha_1(m)}}{\alpha_1(m)}\,dm\bigg)\bigg(\int_{\mathcal{D}} \frac{1}{2}\Big(|\nabla\psi_k|^{\alpha_1(m)-2} - |\nabla\psi|^{\alpha_1(m)-2}\Big)\big(|\nabla\psi_k|^2 - |\nabla\psi|^2\big)\,dm\bigg) \\
&\quad + \mathcal{R}_1\bigg(\int_{\mathcal{D}} \frac{|\nabla\psi|^{\alpha_1(m)} + |\psi|^{\alpha_1(m)}}{\alpha_1(m)}\,dm\bigg)\bigg(\int_{\mathcal{D}} \frac{1}{2}\Big(|\psi_k|^{\alpha_1(m)-2} - |\psi|^{\alpha_1(m)-2}\Big)\big(|\psi_k|^2 - |\psi|^2\big)\,dm\bigg) \geq 0.
\end{aligned}$$

From this inequality and (3.1), $\nabla\psi_k(m) \to \nabla\psi(m)$ and $\psi_k(m) \to \psi(m)$ for a.e. $m \in \mathcal{D}$. We use Fatou's lemma to obtain

$$\underline{\lim_{k\to\infty}} \int_{\mathcal{D}} \frac{\big(|\nabla\psi_k|^{\alpha_1(m)} + |\psi_k|^{\alpha_1(m)}\big)}{\alpha_1(m)}\,dm \geq \int_{\mathcal{D}} \frac{\big(|\nabla\psi|^{\alpha_1(m)} + |\psi|^{\alpha_1(m)}\big)}{\alpha_1(m)}\,dm. \tag{3.2}$$

Similarly, we can derive

$$\underline{\lim_{k\to\infty}} \int_{\mathcal{D}} \frac{\big(|\nabla\varphi_k|^{\alpha_2(m)} + |\varphi_k|^{\alpha_2(m)}\big)}{\alpha_2(m)}\,dm \geq \int_{\mathcal{D}} \frac{\big(|\nabla\varphi|^{\alpha_2(m)} + |\varphi|^{\alpha_2(m)}\big)}{\alpha_2(m)}\,dm. \tag{3.3}$$

However, we also have

$$\begin{aligned}
\lim_{k\to\infty} \big\langle \mathcal{S}_{\alpha_1}(\psi_k), \psi_k-\psi\big\rangle &+ \big\langle \mathcal{S}_{\alpha_2}(\varphi_k), \varphi_k-\varphi\big\rangle \\
&= \lim_{k\to\infty} \big\langle \mathcal{S}(\psi_k,\varphi_k), (\psi_k,\varphi_k) - (\psi,\varphi)\big\rangle \\
&= \lim_{k\to\infty} \big\langle \mathcal{S}(\psi_k,\varphi_k) - \mathcal{S}(\psi,\varphi), (\psi_k,\varphi_k) - (\psi,\varphi)\big\rangle = 0.
\end{aligned}$$

Applying Young's inequality, it can be observed that

$$\begin{aligned}
&\big\langle \mathcal{S}_{\alpha_1}(\psi_k), \psi_k-\psi\big\rangle + \big\langle \mathcal{S}_{\alpha_2}(\varphi_k), \varphi_k-\varphi\big\rangle \\
&\qquad = \mathcal{R}_1\bigg(\int_{\mathcal{D}} \frac{|\nabla\psi_k|^{\alpha_1(m)} + |\psi_k|^{\alpha_1(m)}}{\alpha_1(m)}\,dm\bigg)
\end{aligned}$$

$$
\begin{aligned}
&\times\Big(\int\limits_{\mathcal{D}}\left(|\nabla\psi_k|^{\alpha_1(m)}+|\psi_k|^{\alpha_1(m)}\right)dm-\int\limits_{\mathcal{D}}\Big(|\nabla\psi_k|^{\alpha_1(m)-2}\nabla\psi_k\cdot\nabla\psi+|\psi_k|^{\alpha_1(m)-2}\psi_k\psi\Big)\,dm\Big)\\
&\qquad\qquad+\mathcal{R}_2\Big(\int\limits_{\mathcal{D}}\frac{|\nabla\varphi_k|^{\alpha_2(m)}+|\varphi_k|^{\alpha_2(m)}}{\alpha_2(m)}\,dm\Big)\\
&\times\Big(\int\limits_{\mathcal{D}}\left(|\nabla\varphi_k|^{\alpha_2(m)}+|\varphi_k|^{\alpha_2(m)}\right)dm-\int\limits_{\mathcal{D}}\Big(|\nabla\varphi_k|^{\alpha_2(m)-2}\nabla\varphi_k.\nabla\varphi+|\varphi_k|^{\alpha_2(m)-2}\varphi_k\varphi\Big)\,dm\Big)\\
&\geq\mathcal{R}_1\Big(\int\limits_{\mathcal{D}}\frac{|\nabla\psi_k|^{\alpha_1(m)}+|\psi_k|^{\alpha_1(m)}}{\alpha_1(m)}\,dm\Big)\\
&\qquad\times\Big(\int\limits_{\mathcal{D}}\frac{|\nabla\psi_k|^{\alpha_1(m)}+|\psi_k|^{\alpha_1(m)}}{\alpha_1(m)}\,dm-\int\limits_{\mathcal{D}}\frac{|\nabla\psi|^{\alpha_1(m)}+|\psi|^{\alpha_1(m)}}{\alpha_1(m)}\,dm\Big)\\
&\quad+\mathcal{R}_2\Big(\int\limits_{\mathcal{D}}\frac{|\nabla\varphi_k|^{\alpha_2(m)}+|\varphi_k|^{\alpha_2(m)}}{\alpha_2(m)}\,dm\Big)\\
&\qquad\times\Big(\int\limits_{\mathcal{D}}\frac{|\nabla\varphi_k|^{\alpha_2(m)}+|\varphi_k|^{\alpha_2(m)}}{\alpha_2(m)}\,dm-\int\limits_{\mathcal{D}}\frac{|\nabla\varphi|^{\alpha_2(m)}+|\varphi|^{\alpha_2(m)}}{\alpha_2(m)}\,dm\Big)\\
&\geq\frac{a_1}{(\alpha_1^+)^{c_1^--1}}\Big(\int\limits_{\mathcal{D}}\left(|\nabla\psi_k|^{\alpha_1(m)}+|\psi_k|^{\alpha_1(m)}\right)dm\Big)^{c_1^--1}\\
&\qquad\times\Big(\int\limits_{\mathcal{D}}\frac{|\nabla\psi_k|^{\alpha_1(m)}+|\psi_k|^{\alpha_1(m)}}{\alpha_1(m)}\,dm-\int\limits_{\mathcal{D}}\frac{|\nabla\psi|^{\alpha_1(m)}+|\psi|^{\alpha_1(m)}}{\alpha_1(m)}\,dm\Big)\\
&\quad+\frac{b_1}{(\alpha_2^+)^{c_2^--1}}\Big(\int\limits_{\mathcal{D}}|\nabla\varphi_k|^{\alpha_2(m)}+|\varphi_k|^{\alpha_2(m)}\,dm\Big)^{c_2^--1}\\
&\qquad\times\Big(\int\limits_{\mathcal{D}}\frac{|\nabla\varphi_k|^{\alpha_2(m)}+|\varphi_k|^{\alpha_2(m)}}{\alpha_2(m)}\,dm-\int\limits_{\mathcal{D}}\frac{|\nabla\varphi|^{\alpha_2(m)}+|\varphi|^{\alpha_2(m)}}{\alpha_2(m)}\,dm\Big).
\end{aligned}
$$

Combining (3.2) and (3.3), we get

$$\lim_{k\to\infty}\int\limits_{\mathcal{D}}\frac{\left(|\nabla\psi_k|^{\alpha_1(m)}+|\psi_k|^{\alpha_1(m)}\right)}{\alpha_1(m)}\,dm=\int\limits_{\mathcal{D}}\frac{\left(|\nabla\psi|^{\alpha_1(m)}+|\psi|^{\alpha_1(m)}\right)}{\alpha_1(m)}\,dm$$

and

$$\lim_{k\to\infty}\int\limits_{\mathcal{D}}\frac{\left(|\nabla\varphi_k|^{\alpha_2(m)}+|\varphi_k|^{\alpha_2(m)}\right)}{\alpha_2(m)}\,dm=\int\limits_{\mathcal{D}}\frac{\left(|\nabla\varphi|^{\alpha_2(m)}+|\varphi|^{\alpha_2(m)}\right)}{\alpha_2(m)}\,dm.$$

Then

$$\lim_{n\to\infty}\int\limits_{\mathcal{D}}\left(|\nabla\psi_k|^{\alpha_1(m)}+|\psi_k|^{\alpha_1(m)}\right)dm=\int\limits_{\mathcal{D}}\left(|\nabla\psi|^{\alpha_1(m)}+|\psi|^{\alpha_1(m)}\right)dm$$

and

$$\lim_{n\to\infty}\int\limits_{\mathcal{D}}\left(|\nabla\varphi_k|^{\alpha_2(m)}+|\varphi_k|^{\alpha_2(m)}\right)dm=\int\limits_{\mathcal{D}}\left(|\nabla\varphi|^{\alpha_2(m)}+|\varphi|^{\alpha_2(m)}\right)dm.$$

Applying a technique similar to that presented in [21], we find that

$$\lim_{n\to\infty}\int\limits_{\mathcal{D}}\Big(|\nabla\psi_k-\nabla\psi|^{\alpha_1(m)}+|\psi_k-\psi|^{\alpha_1(m)}\Big)\,dm=0$$

and

$$\lim_{n\to\infty} \int_{\mathcal{D}} \Big(|\nabla\varphi_k - \nabla\varphi|^{\alpha_2(m)} + |\varphi_k - \varphi|^{\alpha_2(m)} \Big)\, dm = 0.$$

Therefore, $(\psi_k, \varphi_k) \to (\psi, \varphi)$. □

Lemma 3.2. *Under the assumptions* (A_1)–(A_3), *the operator* $\mathcal{C} : \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)} \to (\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)})^*$ *defined by*

$$\begin{aligned}&\big\langle \mathcal{C}(\psi,\varphi), (\vartheta,\zeta)\big\rangle\\ &\quad = -\bigg(\int_{\mathcal{D}} \Big(-\delta_1|\psi|^{p(m)-2}\psi + \lambda_1 f(m,\psi,\nabla\psi)\Big)\vartheta + \Big(-\delta_2|\varphi|^{q(m)-2}\varphi + \lambda_2 g(m,\varphi,\nabla\varphi)\Big)\zeta\, dm\bigg)\end{aligned}$$

is compact.

Proof. Let $\mathcal{X} : \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)} \to L^{\alpha_1'(m)}(\mathcal{D}) \times L^{\alpha_2'(m)}(\mathcal{D})$ be an operator characterized by

$$\mathcal{X}(\psi,\varphi) := (\mathcal{X}_p(\psi), \mathcal{X}_q(\varphi)) + (\mathcal{X}_f(\psi), \mathcal{X}_g(\varphi)),$$

with

$$(\mathcal{X}_p(\psi), \mathcal{X}_q(\varphi)) = \big(\delta_1|\psi|^{p(m)-2}\psi, \delta_2|\varphi|^{q(m)-2}\varphi\big)$$

and

$$(\mathcal{X}_f(\psi), \mathcal{X}_g(\varphi)) = \big(-\lambda_1 f(m,\psi,\nabla\psi), -\lambda_2 g(m,\varphi,\nabla\varphi)\big).$$

We do this by demonstrating that $\mathcal{X}$ is bounded and continuous.

Let $(\psi,\varphi) \in \mathcal{W}_0^{\alpha_1(m),\alpha_1(m)}$, we have

$$\begin{aligned}&|\mathcal{X}(\psi,\varphi)|_{L^{\alpha_1'(m)}\times L^{\alpha_2'(m)}}\\ &\quad = \max\Big(\big|\delta_1|\psi|^{p(m)-2}\psi - \lambda_1 f(m,\psi,\nabla\psi)\big|_{\alpha_1'(m)}, \big|\delta_2|\varphi|^{q(m)-2}\varphi - \lambda_2 g(m,\varphi,\nabla\varphi)\big|_{\alpha_2'(m)}\Big)\\ &\qquad \le \big|\delta_1|\psi|^{p(m)-2}\psi - \lambda_1 f(m,\psi,\nabla\psi)\big|_{\alpha_1'(m)} + \big|\delta_2|\varphi|^{q(m)-2}\varphi - \lambda_2 g(m,\varphi,\nabla\varphi)\big|_{\alpha_2'(m)}.\end{aligned}$$

By (2.5), (2.6), (A_2) and (A_3), we get

$$\begin{aligned}&|\mathcal{X}(\psi,\varphi)|_{L^{\alpha_1'(m)}\times L^{\alpha_2'(m)}}\\ &\quad \le \int_{\mathcal{D}} \big|\delta_1|\psi|^{p(m)-2}\psi - \lambda_1 f(m,\psi,\nabla\psi)\big|^{\alpha_1'(m)}\, dm + \int_{\mathcal{D}} \big|\delta_2|\varphi|^{p(m)-2}\psi - \lambda_2 g(m,\varphi,\nabla\varphi)\big|^{\alpha_2'(m)}\, dm\\ &\quad \le C\big(|\lambda_1|^{\alpha_1'^-} + |\lambda_1|^{\alpha_1'^+}\big)\Big(\rho_{\alpha_1'(m)}(\gamma_1) + \rho_{e(m)}(\psi) + \rho_{e(m)}(\nabla\psi)\Big)\\ &\qquad\qquad + C'\big(|\lambda_2|^{\alpha_2'^-} + |\lambda_2|^{\alpha_2'^+}\big)\Big(\rho_{\alpha_2'(m)}(\gamma_2) + \rho_{z(m)}(\varphi) + \rho_{z(m)}(\nabla\varphi)\Big)\\ &\qquad\qquad + \big(|\delta_1|^{\alpha_1'^-} + |\delta_1|^{\alpha_1'^+}\big)\rho_{l_1(m)}(\psi) + \big(|\delta_2|^{\alpha_2'^-} + |\delta_2|^{\alpha_2'^+}\big)\rho_{l_2(m)}(\varphi) + 4\\ &\quad \le C\Big(|\gamma_1|^{p'^+}_{\alpha_1(m)} + |\psi|^{e^+}_{e(m)} + |\psi|^{e^-}_{e(m)} + |\nabla\psi|^{e^+}_{e(m)} + |\nabla\psi|^{e^-}_{e(m)}\Big)\\ &\qquad\qquad + C'\Big(|\gamma_2|^{p'^+}_{\alpha_2(m)} + |\varphi|^{z^+}_{z(m)} + |\varphi|^{z^-}_{z(m)} + |\nabla\varphi|^{z^+}_{z(m)} + |\nabla\varphi|^{z^-}_{z(m)}\Big)\\ &\qquad\qquad + \big(|\delta_1|^{\alpha_1'^-} + |\delta_1|^{\alpha_1'^+}\big)\big(|\psi|^{l_1^-}_{l_1(m)} + |\psi|^{l_1^+}_{l_1(m)}\big)\\ &\qquad\qquad + \big(|\delta_2|^{\alpha_2'^-} + |\delta_2|^{\alpha_2'^+}\big)\big(|\varphi|^{l_2^-}_{l_2(m)} + |\varphi|^{l_2^+}_{l_2(m)}\big) + 4,\end{aligned}$$

where

$$\begin{gathered} e(m) = (r(m)-1)\alpha_1'(m) < \alpha_1(m),\quad z(m) = (s(m)-1)\alpha_2'(m) < \alpha_2(m),\\ l_1(m) = (p(m)-1)\alpha_1'(m),\quad l_2(m) = (q(m)-1)\alpha_2'(m).\end{gathered}$$

Further, using $L^{\alpha_1(m)} \hookrightarrow L^{e(m)}$, $L^{\alpha_2(m)} \hookrightarrow L^{z(m)}$, $L^{\alpha_1(m)} \hookrightarrow L^{l_1(m)}$, $L^{\alpha_2(m)} \hookrightarrow L^{l_2(m)}$ and (2.9), we find that

$$\begin{aligned}
&|\mathcal{X}(\psi,\varphi)|_{L^{\alpha_1'(m)}\times L^{\alpha_2'(m)}} \\
&\quad\le C\Big(|\gamma_1|^{p'^+}_{\alpha_1(m)} + |\psi|^{e^+}_{1,\alpha_1(m)} + |\psi|^{e^-}_{1,\alpha_1(m)}\Big) + C'\Big(|\gamma_2|^{q'^+}_{\alpha_2(m)} + |\varphi|^{z^+}_{1,\alpha_2(m)} + |\varphi|^{z^-}_{1,\alpha_2(m)}\Big) \\
&\qquad\qquad + C''\big(|\psi|^{l_1^-}_{1,\alpha_1(m)} + |\psi|^{l_1^+}_{1,\alpha_1(m)}\big) + C'''\big(|\varphi|^{l_2^-}_{1,\alpha_2(m)} + |\varphi|^{l_2^+}_{1,\alpha_2(m)}\big) + 4 \\
&\quad\le C_{\max}\Big(|\gamma_1|^{p'^+}_{\alpha_1(m)} + |\psi|^{e^+}_{1,\alpha_1(m)} + |\psi|^{e^-}_{1,\alpha_1(m)} + |\gamma_2|^{q'^+}_{\alpha_2(m)} + |\varphi|^{z^+}_{1,\alpha_2(m)} \\
&\qquad\qquad + |\varphi|^{z^-}_{1,\alpha_2(m)} + |\psi|^{l_1^-}_{1,\alpha_1(m)} + |\psi|^{l_1^+}_{1,\alpha_1(m)} + |\varphi|^{l_2^-}_{1,\alpha_2(m)} + |\varphi|^{l_2^+}_{1,\alpha_2(m)}\Big) + 4,
\end{aligned}$$

according to this, $\mathcal{X}$ is bounded on $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})$.

Furthermore, we will demonstrate the continuity of $\mathcal{X}$. Let $(\psi_n,\varphi_n) \to (\psi,\varphi)$ in $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})$, therefore, there exists a subsequence (ψ_k,φ_k) of (ψ_n,φ_n) and (ϱ_1,ϱ_2) in $L^{\alpha_1(m)}(\mathcal{D}) \times L^{\alpha_2(m)}(\mathcal{D})$, (ϱ_3,ϱ_4) in $(L^{\alpha_1(m)}(\mathcal{D}) \times L^{\alpha_2(m)}(\mathcal{D}))^N$ with

$$\begin{gathered}
\psi_k(m) \to \psi(m),\ \ \varphi_k(m) \to \varphi(m) \ \text{ and } \ \nabla\psi_k(m) \to \nabla\psi(m),\ \ \nabla\varphi_k(m) \to \nabla\varphi(m), \\
|\psi_k(m)| \le \varrho_1(m),\ \ |\varphi_k(m)| \le \varrho_2(m) \ \text{ and } \ |\nabla\psi_k(m)| \le |\varrho_3(m)|,\ \ |\nabla\varphi_k(m)| \le |\varrho_4(m)|.
\end{gathered} \tag{3.4}$$

So, by (A_1), passing $k \longrightarrow \infty$, we obtain

$$\begin{aligned}
&\Big(f(m,\psi_k(m),\nabla\psi_k(m)), g(m,\varphi_k(m),\nabla\varphi_k(m))\Big) \\
&\qquad\qquad\qquad \longrightarrow \Big(f\big(m,\psi(m),\nabla\psi(m)\big), g\big(m,\varphi(m),\nabla\varphi(m)\big)\Big)
\end{aligned} \tag{3.5}$$

a.e. $m \in \mathcal{D}$.

Furthermore, according to (A_2), (A_3) and (3.4), we get

$$\begin{aligned}
\big|f(m,\psi_k(m),\nabla\psi_k(m))\big| &\le \beta_1\Big(\gamma_1(m) + |\varrho_1(m)|^{r(m)-1} + |\varrho_3(m)|^{r(m)-1}\Big), \\
\big|g(m,\varphi_k(m),\nabla\varphi_k(m))\big| &\le \beta_2\Big(\gamma_2(m) + |\varrho_2(m)|^{s(m)-1} + |\varrho_4(m)|^{s(m)-1}\Big).
\end{aligned}$$

We know that

$$\begin{aligned}
&\gamma_1 + |\varrho_1|^{r(m)-1} + |\varrho_3(m)|^{r(m)-1} \in L^{\alpha_1(m)}(\mathcal{D}), \\
&\gamma_2 + |\varrho_2|^{s(m)-1} + |\varrho_4(m)|^{s(m)-1} \in L^{\alpha_2(m)}(\mathcal{D})
\end{aligned}$$

and

$$\begin{aligned}
\rho_{\alpha_1(m)}\big(\mathcal{X}_f(\psi_k) - \mathcal{X}_f(\psi)\big) &= \int_{\mathcal{D}} \Big|\lambda_1 f\big(m,\psi_k(m),\nabla\psi_k(m)\big) - \lambda_1 f\big(m,\psi(m),\nabla\psi(m)\big)\Big|^{\alpha_1(m)}\, dm, \\
\rho_{\alpha_2(m)}\big(\mathcal{X}_g(\varphi_k) - \mathcal{X}_g(\varphi)\big) &= \int_{\mathcal{D}} \Big|\lambda_2 g\big(m,\varphi_k(m),\nabla\varphi_k(m)\big) - \lambda_2 g\big(m,\varphi(m),\nabla\varphi(m)\big)\Big|^{\alpha_2(m)}\, dm.
\end{aligned}$$

Subsequently, using Lebesgue's theorem and equation (2.4), we obtain

$$\mathcal{X}_f\psi_k \to \mathcal{X}_f\psi \ \text{ in } \ L^{\alpha_1(m)}(\mathcal{D})$$

and

$$\mathcal{X}_g\varphi_k \to \mathcal{X}_g\varphi \ \text{ in } \ L^{\alpha_2(m)}(\mathcal{D}).$$

And since

$$\mathcal{X}_p\psi_k \to \mathcal{X}_p\psi \ \text{ in } \ L^{\alpha_1(m)}(\mathcal{D})$$

and

$$\mathcal{X}_q\psi_k \to \mathcal{X}_q\psi \ \text{ in } \ L^{\alpha_2(m)}(\mathcal{D}),$$

$\mathcal{X}$ is continuous.

Moreover, let $I^* : L^{\alpha_1(m)}(\mathcal{D}) \times L^{\alpha_2(m)}(\mathcal{D}) \to (\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}))^*$ be the adjoint operator for the embedding of

$$I : \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}) \to L^{\alpha_1(m)}(\mathcal{D}) \times L^{\alpha_2(m)}(\mathcal{D}).$$

We then define

$$I^* \circ \mathcal{X} : \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}) \to (\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}))^*,$$

this is clearly defined by (A_2) and (A_3).

As the embedding operator I is compact, it is well-known that its adjoint operator I^* is also compact. Therefore, $I^* \circ \mathcal{X}$ is compact. □

The main result of this study is the following theorem.

Theorem 3.1. *Under the assumptions (A_1)–(A_3) and (M_0), system (1.1) has a weak solution (ψ, φ) in $\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}$.*

Proof. First, $(\psi, \varphi) \in \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})$ is a weak solution of (1.1) if and only if

$$\mathcal{S}(\psi, \varphi) = -\mathcal{C}(\psi, \varphi), \tag{3.6}$$

the operators

$$\mathcal{S} : \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}) \to (\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}))^*$$

and

$$\mathcal{C} : \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}) \to (W_0^{\alpha_1(m)}(\mathcal{D}) \times W_0^{\alpha_2(m)}(\mathcal{D}))^*$$

are defined in Lemmas 3.1 and 3.2, respectively.

Due to the properties of the operator $\mathcal{S}$ described in Lemma 3.1, and taking into account the Minty–Browder Theorem (refer to [41, Theorem 26 A]), the inverse operator

$$\mathcal{Y} := \mathcal{S}^{-1} : (\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}))^* \to \mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D})$$

is bounded, continuous and of class (S_+).

Additionally, by Lemma 3.2, it can be established that the operator $\mathcal{C}$ is bounded, continuous and quasimonotone. Then (3.6) is equivalent to

$$(\psi, \varphi) = \mathcal{Y}(\vartheta, \zeta) \ \text{ and } \ (\vartheta, \zeta) + (\mathcal{C} \circ \mathcal{Y})(\vartheta, \zeta) = 0, \ \ (\vartheta, \zeta) \in (\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}))^*. \tag{3.7}$$

We will use the topological degree theory to solve (3.7). Toward this end, we set

$$\mathcal{Q} := \Big\{(\vartheta, \zeta) \in (\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}))^* : \ \ \exists\, t \in [0,1] \ \text{ such that } \ (\vartheta, \zeta) + t(\mathcal{C} \circ \mathcal{Y})(\vartheta, \zeta) = 0\Big\}.$$

Subsequently, we establish that $\mathcal{Q}$ is bounded in $(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}))^*$.

Let $(\vartheta, \zeta) \in \mathcal{Q}$ and put $(\psi, \varphi) := \mathcal{Y}(\vartheta, \zeta)$, then

$$\|\mathcal{Y}(\vartheta, \zeta)\|_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}} = \|(\psi, \varphi)\|_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}} = \max\big\{\|\psi\|_{\alpha_1(m)}, \|\varphi\|_{\alpha_2(m)}\big\},$$

by taking into account that $\|\cdot\|_{\alpha_1(m)} = |\nabla \cdot|_{\alpha_1(m)}$.

Now, we distinguish two cases and related subcases as follows:

Case 1: $\|\mathcal{Y}(\vartheta, \zeta)\|_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}} = \|\psi\|_{\alpha_1(m)} = |\nabla\psi|_{\alpha_1(m)}$.

Case 1.1: If $|\nabla\psi|_{\alpha_1(m)} \leq 1$, then $\|\mathcal{Y}(\vartheta, \zeta)\|_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}}$ is bounded.

Case 1.2: If $|\nabla\psi|_{\alpha_1(m)} > 1$, then using (2.2), (A_2), (2.1), (2.6) and the Young inequality, we obtain

$$\begin{aligned}
\|\mathcal{Y}(\vartheta,\zeta)\|^{\alpha_1^-} &= |\nabla\psi|^{\alpha_1-}_{\alpha_1(m)} \leq \rho_{\alpha_1(m)}(\nabla\psi) \\
&\leq \langle \mathcal{S}\psi,\psi\rangle = \langle \vartheta, \mathcal{Y}\vartheta\rangle = -t\big\langle (\mathcal{C}\circ\mathcal{Y})\vartheta, \mathcal{Y}\vartheta\big\rangle = t\int_{\mathcal{D}} \Big(-\delta_1|\psi|^{p(m)-2}\psi + \lambda_1 f(x,\psi,\nabla\psi)\Big)\psi\, dm \\
&\leq C_{\max}\Big(\rho_{p(m)}(\psi) + \int_{\mathcal{D}} |\gamma_1(m)\psi(m)|\, dm + \rho_{r(m)}(\psi) + \int_{\mathcal{D}} |\nabla\psi|^{r(m)-1}|\psi|\, dm\Big) \\
&\leq C_{\max}\Big(|\psi|^{p^-}_{p(m)} + |\psi|^{p^-}_{p(m)} + |\gamma_1|_{\alpha_1(m)}|\psi|_{\alpha_1(m)} + |\psi|^{r^+}_{r(m)} + |\psi|^{r^-}_{r(m)} + \frac{1}{r'^-}\,\rho_{r(m)}(\nabla\psi) + \frac{1}{r-}\,\rho_{r(m)}(\psi)\Big) \\
&\leq C_{\max}\Big(|\psi|^{p^-}_{p(m)} + |\psi|^{p^-}_{p(m)} + |\psi|_{\alpha_1(m)} + |\psi|^{r^+}_{r(m)} + |\psi|^{r^-}_{r(m)} + |\nabla\psi|^{r^+}_{r(m)}\Big).
\end{aligned}$$

By (2.9), $L^{\alpha_1(m)} \hookrightarrow L^{p(m)}$ and $L^{\alpha_1(m)} \hookrightarrow L^{r(m)}$, we obtain

$$\|\mathcal{Y}(\vartheta,\zeta)\|^{\alpha_1^-}_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}} \leq C\Big(|\mathcal{Y}(\vartheta)|^{p^+}_{1,\alpha_1(m)} + |\mathcal{Y}(\vartheta)|_{1,\alpha_1(m)} + |\mathcal{Y}(\vartheta)|^{r^+}_{1,\alpha_1(m)}\Big),$$

hence $\|\mathcal{Y}(\vartheta,\zeta)\|_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}}$ is bounded.

Case 2: $\|\mathcal{Y}(\vartheta,\zeta)\|_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}} = \|\varphi\|_{\alpha_2(m)} = |\nabla\varphi|_{\alpha_2(m)}$.

Case 2.1: If $|\nabla\varphi|_{\alpha_2(m)} \leq 1$, then $\|\mathcal{Y}(\vartheta,\zeta)\|_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}}$ is bounded.

Case 2.2: If $\|\nabla\varphi\|_{\alpha_2(m)} > 1$, then using (2.2), (A_3), (2.1), (2.6) and the Young inequality, we have

$$\|\mathcal{Y}(\vartheta,\zeta)\|^{\alpha_2^-}_{\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}} \leq C\Big(|\mathcal{Y}(\zeta)|^{q^+}_{1,\alpha_2(m)} + |\mathcal{Y}(\zeta)|_{1,\alpha_2(m)} + |\mathcal{Y}(\zeta)|^{s^+}_{1,\alpha_2(m)}\Big).$$

So, we infer that $\{\mathcal{Y}(\vartheta,\zeta):\ (\vartheta,\zeta)\in\mathcal{Q}\}$ is bounded.

Since the operator $\mathcal{C}$ is bounded, and by (3.7), there exists $R>0$ such that

$$\|(\vartheta,\zeta)\|_{(\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}))^*} < R \ \text{ for all } \ (\vartheta,\zeta)\in\mathcal{Q}.$$

Therefore,

$$(\vartheta,\zeta) + t(\mathcal{C}\circ\mathcal{Y})(\vartheta,\zeta) \neq 0 \ \text{ for all } \ (\vartheta,\zeta)\in\partial\mathcal{Q}_R(0) \ \text{ and all } \ t\in[0,1],$$

where $\mathcal{Q}_R(0)$ is the ball of center 0 and radius R in $(\mathcal{W}^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}))^*$.

Moreover, we also have from Lemma 2.1 that

$$I + \mathcal{C}\circ\mathcal{Y} \in \mathcal{T}_{\mathcal{Y}}(\overline{\mathcal{Q}_{\mathrm{R}}(0)}) \ \text{ and } \ I = \mathcal{S}\circ\mathcal{Y} \in \mathcal{T}_{\mathcal{Y}}(\overline{\mathcal{Q}_{\mathrm{R}}(0)}).$$

Following that, we establish the homotopy $\mathcal{H}:[0,1]\times\overline{\mathcal{Q}_R(0)} \to (\mathcal{W}_0^{\alpha_1(m),\alpha_2(m)}(\mathcal{D}))^*$ by

$$\mathcal{H}(t,(\vartheta,\zeta)) := (\vartheta,\zeta) + t(\mathcal{C}\circ\mathcal{Y})(\vartheta,\zeta).$$

Employing the homotopy invariance and normalization properties of degree d as mentioned in Theorem 2.1, we obtain

$$d(I + \mathcal{C}\circ\mathcal{Y}, \mathcal{Q}_R(0), 0) = d(I, \mathcal{Q}_R(0), 0) = 1 \neq 0.$$

Since $d(I+\mathcal{C}\circ\mathcal{Y}, \mathcal{Q}_R(0), 0) \neq 0$, based on the existence property of degree d mentioned in Theorem 2.1, there exists $(\vartheta,\zeta)\in\mathcal{Q}_R(0)$ such that

$$(\vartheta,\zeta) + (\mathcal{C}\circ\mathcal{Y})(\vartheta,\zeta) = 0.$$

At last, we deduce that $(\psi,\varphi) = \mathcal{Y}(\vartheta,\zeta)$ is a weak solution of (1.1). □

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