Memoirs on Differential Equations and Mathematical Physics

Volume 94, 2025, 145–160

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ON THE SOLVABILITY OF A NON-LOCAL PROBLEM WITH INTEGRAL BOUNDARY CONDITION FOR A SECOND ORDER PARABOLIC EQUATION

Abstract. In this paper, the existence and uniqueness of a strong solution of a second order parabolic equation with integral boundary condition is proved. First, we establish a priori estimate and prove that the range of the operator generated by the considered problem is dense. The technique of deriving the a priori estimate is based on the construction of a suitable multiplicator. From the resulted energy estimate, it is possible to establish the solvability of the linear problem.

2020 Mathematics Subject Classification. 35B45, 35G10.

Key words and phrases. Energy inequality, integral boundary conditions, strong solution, second order parabolic equation.

რეზიუმე. ნაშრომში დამტკიცებულია ძლიერი ამონახსნის არსებობა და ერთადერთობა მეორე რიგის პარაბოლური განტოლებისთვის ინტეგრალური სასაზღვრო პირობით. თავდაპირველად დადგენილია აპრიორული შეფასება და დამტკიცებულია, რომ განხილული ამოცანის მიერ გენერირებული ოპერატორის მნიშვნელობათა სიმრავლე მკვრივია. აპრიორული შეფასების მიღების ტექნიკა ეფუძნება შესაფერისი მულტიპლიკატორის აგებას. მიღებული ენერგეტიკული შეფასების საშუალებით შესაძლებელი ხდება წრფივი ამოცანის ამოხსნადობის დადგენა.

1 Introduction and statement of the problem

Some problems related to physical and technical issues can be effectively described in terms of nonlocal problems with integral conditions in partial differential equations. These nonlocal conditions arise mainly when the values on the boundary cannot be measured directly, but their average values are known. Therefore, the investigation of these problems requires a separate study. The importance of problems with integral conditions has been pointed out by Samarskii [20]. These mathematical models are encountered in many engineering models such as heat conduction [2,3], plasma physics [20], thermoelasticity [21], electrochemistry [4], chemical diffusion [5] and underground water flow [16,23]. The first paper devoted to second-order partial differential equations with nonlocal integral conditions goes back to Cannon [3]. This type of boundary value problems, which are combined with Dirichlet or Newmann condition and integral condition, or with purely integral conditions, have been investigated for parabolic equations in [1–4,6,9,10,12–14,24], for hyperbolic equations in [1,18,19], for mixed type equations in [7,8], and elliptic equations with nonlocal conditions were considered by Gushchin and Mikhailov [11], A. L. Skubachevski [22] and Peneiah [17].

In this paper, we prove the existence and uniqueness of a strong solution of a class of nonlocal mixed second order parabolic problems in which nonlocal boundary conditions with integral conditions given only on parts of the boundary are combined. This problem is stated as follows: Let us consider the rectangular domain $\Omega =]0,1[\times]0,T[$ with $T < +\infty$, then the problem is to find a solution $u(x,t)$ of the following non-classical boundary value problem:

$$\mathcal{L}u = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(a(x,t)\,\frac{\partial u}{\partial x}\Big) = f(x,t) \ \text{ for } \ (x,t) \in \Omega =]0,1[\times]0,T[\tag{1.1}$$

with the initial condition

$$lu = u(x,0) = \varphi(x), \ \ \forall\, x \in [0,1], \tag{1.2}$$

the nonlocal boundary condition

$$\frac{\partial u}{\partial x}\,(0,t) = \frac{\partial u}{\partial x}\,(1,t), \ \ \forall\, t \in [0,T], \tag{1.3}$$

and the integral condition

$$\int_0^\alpha u(x,t)\,dx = 0, \ \ \forall\, t \in [0,T], \ \ \text{where } \ 0 < \alpha < 1. \tag{1.4}$$

It is worth mentioning that in [15], the authors proved the existence, uniqueness and continuous dependence of a strong solution in weighted Sobolev spaces to the problem

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(a\,\frac{\partial u}{\partial x}\Big) = f(x,t)$$

with the initial condition

$$lu = u(x,0) = \varphi(x), \ \ \forall\, x \in [0,1],$$

the boundary condition

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(1,t)}{\partial x}\,, \ \ \forall\, t \in [0,T],$$

and the integral condition

$$\int_0^1 u(x,t)\,dx = m(t), \ \ \forall\, t \in [0,T].$$

In addition, we assume that the function $a(x,t)$ and its derivatives satisfy the conditions

$$0 < a_0 \le a(x,t) \le a_1, \quad a_2 \le \frac{\partial a}{\partial t} \le a_3, \quad \Big|\frac{\partial a}{\partial x}\Big| \le b, \ \ \forall\,(x,t) \in \overline{\Omega},$$

where the functions $\varphi(x)$, $f(x,t)$ are given, and we assume that the following matching conditions are satisfied:

$$\frac{\partial\varphi}{\partial x}(0)=\frac{\partial\varphi}{\partial x}(1),\quad \int\limits_0^\alpha \varphi(x)\,dx=0.$$

In the present paper, the motivation is to study and find a solution to the stated problem (1.1)–(1.4) without imposing any condition on the constant α in the interval $[0,1]$. In addition, the linear problem of the parabolic equation with integral condition defined on one part of the boundary is solved.

First, an a priori estimate is established for the linear problem and, using the functional analysis method, the density of the operator range generated by the problem under consideration is proved. The given problem can be considered as finding a solution of the operator equation given by

$$Lu=(\mathcal{L}u,lu)=(f,\varphi)=\mathcal{F},$$

where the operator L has a domain of definition $D(L)$ consisting of functions $u\in L^2(\Omega)$ such that $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}\in L^2(\Omega)$, $\frac{\partial^2 u}{\partial x\partial t}\in L^2(\Omega)$ and satisfying conditions (1.3) and (1.4).

The operator L is defined on E into F, where E is the Banach space of functions $u\in L^2(\Omega)$ with the finite norm

$$\|u\|_E^2=\int\limits_\Omega \theta(x)\Big[\Big|\frac{\partial u}{\partial t}\Big|^2+\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2\Big]\,dx\,dt+\sup_{0\le t\le T}\Big(\int\limits_0^1\theta(x)\Big|\frac{\partial u}{\partial x}\Big|^2\,dx+\int\limits_0^1|u|^2\,dx\Big),$$

F is the Hilbert space of functions $\mathcal{F}=(f,\varphi)$, $f\in L^2(\Omega)$, $\varphi\in H^1(0,1)$ with the finite norm

$$\|\mathcal{F}\|_F^2=\int\limits_0^T\int\limits_0^1\theta(x)|f|^2\,dx+\int\limits_0^1\theta(x)\Big(\Big|\frac{d\varphi}{dx}\Big|^2+|\varphi|^2\Big)\,dx,$$

where

$$\theta(x)=\begin{cases}\dfrac{x^2}{\alpha^2}, & x\in(0,\alpha),\\[2mm] \dfrac{(1-x)^2}{(1-\alpha)^2}, & x\in(\alpha,1).\end{cases}$$

Then we show that the operator L has a closure $\overline{L}$ and later on, in Section 2, we establish an energy inequality of the type

$$\|u\|_E\le k\|Lu\|_F. \tag{1.5}$$

It can be proved in a standard way that the operator $L:E\to F$ is closable. Let $\overline{L}$ be the closure of this operator with the domain of definition $D(\overline{L})$.

Definition. A solution of the operator equation $\overline{L}u=\mathcal{F}$ is called a strong solution of problem (1.1)–(1.4).

The a priori estimate (1.5) can be extended to the strong solution, that is, we have the inequality

$$\|u\|_E\le k\|\overline{L}u\|_F,\ \ \forall\, u\in D(\overline{L}).$$

This last inequality implies the following corollaries.

Corollary 1.1. *If a strong solution of (1.1)–(1.4) exists, it is unique and depends continuously on $F=(f,\,\varphi)$.*

Corollary 1.2. *The range $R(\overline{L})$ of $\overline{L}$ is closed in F and $\overline{R(L)}=R(\overline{L})$.*

Corollary 1.2 shows that to prove that problem (1.1)–(1.4) has a strong solution for arbitrary F, it suffices to prove that the set $R(L)$ is dense in F.

2 Uniqueness and continuous dependence

In this section, we establish an a priori estimate and deduce the uniqueness and continuous dependence of the solution considering the initial statement.

Theorem 2.1. *There exists a positive constant k such that for each function $u \in D(L)$, we have*

$$\|u\|_E \le k\|Lu\|_F.$$

Proof. Let

$$Mu = \begin{cases} \dfrac{\lambda x^2}{2\alpha^2}\,\dfrac{\partial u}{\partial t} - \dfrac{\lambda x}{\alpha^2}\,J_0^x u - \Big(\dfrac{\delta_1}{2}\,x^2 + \dfrac{\lambda}{(1-\alpha)^2 a(0,t)}\Big) J_x^\alpha a e^{\beta(1-\zeta)} J_\alpha^\zeta u \\ \qquad\qquad + \dfrac{\lambda}{\alpha(1-\alpha)^2 a'(0,t)}\,(\alpha - x) e^{\frac{x}{\alpha}} J_0^1 a e^{\beta(1-x)} J_\alpha^x u\,, \quad x \in [0,\alpha], \\ \dfrac{\lambda(x-1)^2}{2(1-\alpha)^2}\,\dfrac{\partial u}{\partial t} + \dfrac{\lambda(1-x)}{(1-\alpha)^2}\,J_\alpha^x u + \Big(\dfrac{\delta}{2}\,(1-x)^2 + \dfrac{\lambda}{(1-\alpha)^2 a(1,t)}\Big) J_\alpha^x a e^{\beta(1-\zeta)} J_\alpha^\zeta u\,, \quad x \in [\alpha, 1], \end{cases}$$

where

$$J_\alpha^x u = \int_\alpha^x \frac{\partial u}{\partial t}\,(\zeta, t)\,d\zeta$$

and

$$\begin{cases} \lambda > 0, \\ \beta \le \min\left(\dfrac{-b}{a_0}, \dfrac{1}{(1-\alpha)}\,\ln\dfrac{a_0}{4a_1}\,\dfrac{(1-\alpha)^2}{\alpha^2}, \dfrac{1}{(1-\alpha)}\,\ln\dfrac{a_0}{4a_1}\,\dfrac{\lambda}{(1-\alpha)^2}\,\dfrac{1}{\frac{\delta}{2}(1-\alpha)^2 + \frac{\lambda}{(1-\alpha)^2 a_0}}\right), \\ \delta_1 < \dfrac{-16\lambda e^2 a_1^2}{\alpha^3(1-\alpha)} - 256\Big(\dfrac{1}{a_0} - \dfrac{1}{a_1}\Big)^2\,\dfrac{\lambda a_1^2}{\alpha^3(1-\alpha)}\,, \\ 0 < -\delta < \dfrac{\lambda}{(1-\alpha)^2 a_1}\,. \end{cases}$$

We consider the quadratic form obtained by multiplying equation (1.1) by $\exp(-ct)\overline{Mu}$ in $L^2(\Omega^s)$ with $\Omega^s = [0,1] \times [0,s]$ and $c > 0$. Taking the real part, we have

$$\begin{aligned} \Phi(u,u) &= \operatorname{Re}\int_{\Omega^s} \exp(-ct) f(x,t)\overline{Mu}\,dx\,dt \\ &= -\operatorname{Re}\int_{\Omega^s} \exp(-ct)\,\frac{\partial}{\partial x}\Big(a(x,t)\,\frac{\partial u}{\partial x}\Big)\overline{Mu}\,dx\,dt + \operatorname{Re}\int_{\Omega^s} \exp(-ct)\,\frac{\partial u}{\partial t}\,\overline{Mu}\,dx\,dt. \end{aligned} \tag{2.1}$$

Mu by its expression in the right-hand side of (2.1), integrating by parts with respect to x and to t, and using conditions (1.2)–(1.4), we obtain

$$\begin{aligned} &\operatorname{Re}\int_0^s\int_0^1 e^{-ct} f\overline{Mu}\,dx\,dt \\ &\qquad = \frac{\lambda}{2\alpha^2}\int_0^s\int_0^\alpha x^2 e^{-ct}\Big|\frac{\partial u}{\partial t}\Big|^2\,dx\,dt + \frac{\lambda}{2(1-\alpha)^2}\int_0^s\int_\alpha^1 (1-x)^2 e^{-ct}\Big|\frac{\partial u}{\partial t}\Big|^2\,dx\,dt \\ &+ \frac{\lambda}{4\alpha^2}\int_0^s\int_0^\alpha x^2\Big(ca - \frac{\partial a}{\partial t}\Big)e^{-ct}\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dt + \frac{\lambda}{4(1-\alpha)^2}\int_0^s\int_\alpha^1 (1-x)^2\Big(ca - \frac{\partial a}{\partial t}\Big)e^{-ct}\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dt \end{aligned}$$

$$+\frac{1}{2}\int_0^s\int_0^\alpha\Big(ca-\frac{\partial a}{\partial t}\Big)\Big(\frac{\lambda}{\alpha^2}-\Big(\frac{\delta_1}{2}x^2+\frac{\lambda}{(1-\alpha)^2a(0,t)}\Big)a(x,t)e^{\beta(1-x)}\Big)e^{-ct}|u|^2\,dx\,dt$$

$$+\frac{1}{2}\int_0^s\int_\alpha^1\Big[-\delta(1-x)ae^{\beta(1-x)}$$

$$+\Big(\frac{\delta}{2}(1-x)^2+\frac{\lambda}{(1-\alpha)^2a(1,t)}\Big)(a_x-\beta a)e^{\beta(1-x)}\Big]e^{-ct}\int_\alpha^x\Big(ca-\frac{\partial a}{\partial t}\Big)|u|^2\,d\zeta\,dx\,dt$$

$$+\int_0^s\int_0^\alpha\Big(\frac{\lambda}{2\alpha^2}-\Big(\frac{\delta_1}{2}x^2+\frac{\lambda}{(1-\alpha)^2a(0,t)}\Big)ae^{\beta(1-x)-ct}\Big)\Big|\int_\alpha^x\frac{\partial u}{\partial t}\Big|^2\,d\zeta\,dx\,dt$$

$$+\int_0^s\Big(\frac{\lambda}{2(1-\alpha)^2}-\Big(\frac{\delta}{2}\,(1-\alpha)^2+\frac{\lambda}{(1-\alpha)^2a(1,t)}\Big)a(\alpha,t)e^{\beta(1-\alpha)}\Big)\int_\alpha^1\Big|\int_\alpha^x\frac{\partial u}{\partial t}\,d\zeta\Big|^2\,dx\,dt$$

$$+\int_0^s\int_\alpha^1\Big(-\delta(1-x)a(x,t)e^{\beta(1-x)}+\Big(\frac{\delta}{2}\,(1-x)^2+\frac{\lambda}{(1-\alpha)^2a(1,t)}\Big)(a_x-\beta a)e^{\beta(1-x)}\Big)\int_\alpha^x\Big|\int_\alpha^\eta\frac{\partial u}{\partial t}\,d\zeta\Big|^2\,dx\,dt$$

$$+\frac{\lambda}{4\alpha^2}\int_0^\alpha\frac{x^2}{2}\,ae^{-ct}\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\Big|_{t=s}+\frac{\lambda}{4(1-\alpha)^2}\int_\alpha^1(1-x)^2ae^{-ct}\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\Big|_{t=s}$$

$$+\frac{1}{2}\int_0^\alpha\Big(\frac{\lambda}{\alpha^2}-\Big(\frac{\lambda}{(1-\alpha)^2a(0,t)}+\frac{\delta_1}{2}x^2\Big)a(x,t)e^{\beta(1-x)}\Big)ae^{-ct}|u|^2\,dx\Big|_{t=s}$$

$$+\frac{1}{2}\int_\alpha^1\Big[-\delta(1-x)ae^{\beta(1-x)}+\Big(\frac{\delta}{2}\,(1-x)^2+\frac{\lambda}{(1-\alpha)^2a(1,t)}\Big)(a_x-\beta a)e^{\beta(1-x)}\Big]e^{-ct}\int_\alpha^x a|u|^2\,dx\,dt\Big|_{t=s}$$

$$-\frac{\lambda}{4\alpha^2}\int_0^\alpha\frac{x^2}{2}\,a(x,0)\Big|\frac{d\varphi}{dx}\Big|^2\,dx-\frac{\lambda}{4(1-\alpha)^2}\int_\alpha^1(1-x)^2a(x,0)\Big|\frac{d\varphi}{dx}\Big|^2\,dx$$

$$-\frac{1}{2}\int_0^\alpha\Big(\frac{\lambda}{\alpha^2}-\Big(\frac{\lambda}{(1-\alpha)^2a(0,t)}+\frac{\delta_1}{2}\,x^2\Big)a(x,0)e^{\beta(1-x)}\Big)a(x,0)|\varphi|^2\,dx$$

$$-\frac{1}{2}\int_\alpha^1\Big[\frac{\lambda}{(1-\alpha)^2}-\Big(\frac{\lambda}{(1-\alpha)^2a(1,0)}+\frac{\delta}{2}\,(1-x)^2\Big)a(x,0)e^{\beta(1-x)}\Big]a(x,0)|\varphi|^2\,dx$$

$$+\frac{1}{2}\int_0^s\int_0^\alpha\Big[\frac{-\lambda a^2\,\frac{\partial a}{\partial t}\,(0,t)}{(1-\alpha)^2a^2(0,t)}+\Big(\frac{\delta_1}{2}x^2+\frac{\lambda}{(1-\alpha)^2a(0,t)}\Big)a\,\frac{\partial a}{\partial t}\Big]e^{\beta(1-x)-ct}|u|^2\,dx\,dt$$

$$-\frac{1}{2}\int_0^s\int_\alpha^1\Big[\frac{\lambda a\,\frac{\partial a}{\partial t}\,(1,t)}{(1-\alpha)^2a^2(1,t)}-\Big(\frac{\delta}{2}\,(1-x)^2+\frac{\lambda}{(1-\alpha)^2a(1,t)}\Big)\,\frac{\partial a}{\partial t}\Big]ae^{\beta(1-x)-ct}|u|^2\,dx\,dt$$

$$-\delta_1\,\mathrm{Re}\int_0^s\int_0^\alpha xe^{-ct}\int_\alpha^x\frac{\partial u}{\partial t}\,d\zeta\;J_x^\alpha ae^{\beta(1-\zeta)}\overline{J_\alpha^\zeta u}\,d\zeta\,dx\,dt$$

$$+\frac{\lambda}{\alpha^2(1-\alpha)^2}\,\mathrm{Re}\int_0^s\int_0^\alpha xe^{\frac{x}{\alpha}-ct}\int_0^x\frac{\partial u}{\partial t}\,dx\;J_0^1ae^{\beta(1-x)}\overline{J_\alpha^x}u\,d\zeta\,dx\,dt$$

$$+\,\delta\,\mathrm{Re}\int_0^s\int_\alpha^1 (1-x)e^{-ct}\int_\alpha^x \frac{\partial u}{\partial t}\,d\zeta\int_\alpha^x ae^{\beta(1-\zeta)}\overline{\int_\alpha^\zeta \frac{\partial u}{\partial t}\,d\zeta}\;dx\,dt$$

$$-\,\frac{\lambda}{\alpha^2(1-\alpha)^2}\,\mathrm{Re}\int_0^s\int_0^\alpha x\,\frac{a(x,t)}{a(0,t)}\,e^{\frac{x}{\alpha}-ct}\,\frac{\partial u}{\partial x}\,\overline{\int_0^1 ae^{\beta(1-x)}J_\alpha^x u\,d\zeta}\;dx\,dt$$

$$+\,\frac{\lambda}{\alpha^2}\,\mathrm{Re}\int_0^s\int_0^\alpha \frac{\partial a}{\partial x}\,e^{-ct}u\overline{J_0^x u}\,dx\,dt-\mathrm{Re}\int_0^s\int_0^\alpha \delta_1 xae^{-ct}\,\frac{\partial u}{\partial x}\,\overline{\int_x^\alpha ae^{\beta(1-\zeta)}J_\alpha^\zeta u\,d\zeta}\;dx\,dt$$

$$+\,\frac{\lambda}{(1-\alpha)^2}\,\mathrm{Re}\int_0^s\int_\alpha^1 \frac{\partial a}{\partial x}\,e^{-ct}u\overline{J_x^\alpha u}\,dx\,dt-\delta\,\mathrm{Re}\int_0^s\int_\alpha^1 (1-x)a(x,t)e^{-ct}\,\frac{\partial u}{\partial x}\int_\alpha^x ae^{\beta(1-\zeta)}\overline{J_\alpha^\zeta u}\,d\zeta\,dx\,dt$$

$$-\,\mathrm{Re}\int_0^s\int_0^\alpha \Big(\delta_1 xa^2+\Big(\frac{\delta_1}{2}x^2+\frac{\lambda}{(1-\alpha)^2a(0,t)}\Big)\Big(2a\,\frac{\partial a}{\partial x}-\beta a^2\Big)\Big)e^{\beta(1-x)-ct}u\overline{J_\alpha^x u}\,dx\,dt$$

$$-\,\mathrm{Re}\int_0^s\int_\alpha^1 \Big[-\delta(1-x)a^2+\Big(\frac{\delta}{2}(1-x)^2+\frac{\lambda}{(1-\alpha)^2a(1,t)}\Big)\Big(2a\,\frac{\partial a}{\partial x}-\beta a^2\Big)\Big]e^{\beta(1-x)-ct}u\overline{J_\alpha^\zeta u}\,d\eta\,dt$$

$$+\,\frac{\lambda}{(1-\alpha)^2a(1,t)}\,\mathrm{Re}\int_0^s e^{-ct}\int_\alpha^1 \frac{\partial u}{\partial t}\,dx\int_\alpha^1 ae^{\beta(1-\zeta)}\overline{J_\alpha^\zeta u}\,dx\,dt. \tag{2.2}$$

Using the fact that

$$\int_0^s\int_0^\alpha e^{-ct}\Big|\int_\alpha^x \frac{\partial u}{\partial t}\Big|^2\,dx\,dt\le 4\int_0^s\int_0^\alpha x^2e^{-ct}\Big|\frac{\partial u}{\partial t}\Big|^2\,dx\,dt,$$

$$\int_0^s\int_\alpha^1 e^{-ct}\Big|\int_\alpha^x \frac{\partial u}{\partial t}\Big|^2\,dx\,dt\le 4\int_0^s\int_\alpha^1 (1-x)^2e^{-ct}\Big|\frac{\partial u}{\partial t}\Big|^2\,dx\,dt,$$

$$\int_0^s\int_\alpha^1 e^{-ct}|u|^2\,dx\,dt\le 4\int_0^s\int_\alpha^1 (1-x)^2e^{-ct}\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dt$$
$$+\,\frac{2(1-\alpha)}{\alpha}\int_0^s\int_0^\alpha x^2e^{-ct}\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dt+\frac{4(1-\alpha)}{\alpha}\int_0^s\int_0^\alpha e^{-ct}|u|^2\,dx\,dt$$

and the ε-inequalities in the last twelve terms in (2.2), we get

$$\frac{\lambda}{64\alpha^2}\int_0^s\int_0^\alpha x^2e^{-ct}\Big|\frac{\partial u}{\partial t}\Big|^2\,dx\,dt+\frac{\lambda}{128(1-\alpha)^2}\int_0^s\int_\alpha^1 (1-x)^2e^{-ct}\Big|\frac{\partial u}{\partial t}\Big|^2\,dx\,dt$$
$$+\Big(c-\frac{a_3}{a_0}-m_1\Big)\int_0^s\int_0^\alpha x^2e^{-ct}\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dt+\Big(c-\frac{a_3}{a_0}-m_2\Big)\int_0^s\int_\alpha^1 (1-x)^2e^{-ct}\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dt$$
$$+\,\frac{\lambda}{4\alpha^2}\,(ca_0-a_3)-\max\{m_3,m_4,m_5\}\int_0^s\int_0^\alpha e^{-ct}|u|^2\,dx\,dt$$
$$+\,\frac{\lambda}{4(1-\alpha)^2}\,(ca_0-a_3)(-\beta a_0-b)\int_\alpha^1\int_\alpha^x e^{-ct}|u|^2\,dx\,dt$$

$$+\int_0^s\int_\alpha^1\Big(-\delta(1-x)a(x,t)e^{\beta(1-x)}+\Big(\frac{\delta}{2}\,(1-x)^2+\frac{\lambda}{(1-\alpha)^2a(1,t)}\Big)(a_x-\beta a)e^{\beta(1-x)}\Big)e^{-ct}\int_\alpha^x\Big|\int_\alpha^\eta\frac{\partial u}{\partial t}\,d\zeta\Big|^2\,dx\,dt$$

$$+\frac{\lambda}{2\alpha^2}\int_0^s\int_0^\alpha e^{-ct}\Big|\int_\alpha^x\frac{\partial u}{\partial t}\Big|^2\,dx\,dt+\frac{\lambda}{4(1-\alpha)^2}\int_0^s\int_\alpha^1 e^{-ct}\Big|\int_\alpha^x\frac{\partial u}{\partial t}\,d\zeta\Big|^2\,dx\,dt$$

$$+\int_0^\alpha\frac{x^2}{2}\,ae^{-ct}\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\Big|_{t=s}+\int_\alpha^1(1-x)^2\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\Big|_{t=s}+\int_0^\alpha|u|^2\,dx\Big|_{t=s}+\int_\alpha^1|u|^2\,dx\Big|_{t=s}$$

$$\leq-\operatorname{Re}\int_0^s\int_0^1 e^{-ct}f\overline{Mu}\,dx\,dt-\operatorname{Re}\int_0^s\frac{\lambda}{(1-\alpha)^2a(1,t)}e^{-ct}\int_\alpha^1\frac{\partial u}{\partial t}\,dx\int_\alpha^1 ae^{\beta(1-x)}\overline{J_\alpha^x u}\,dx\,dt$$

$$+\frac{\lambda a_1}{4\alpha^2}\int_0^\alpha\frac{x^2}{2}\,\Big|\frac{d\varphi}{dx}\Big|^2\,dx+\frac{\lambda a_1}{4(1-\alpha)^2}\int_\alpha^1(1-x)^2\Big|\frac{d\varphi}{dx}\Big|^2\,dx$$

$$+\frac{1}{2}\left(\frac{\lambda}{\alpha^2}-\Big(\frac{\lambda a_0e^\beta}{(1-\alpha)^2a_1}+\frac{\delta_1}{2}\,a_1\alpha^2e^{\beta(1-\alpha)}\Big)\right)\int_0^\alpha|\varphi|^2\,dx$$

$$+\frac{1}{2}\left(\frac{\lambda}{(1-\alpha)^2}-\Big(\frac{\delta}{2}a_1(1-\alpha)^2+\frac{\lambda a_0}{(1-\alpha)^2a_1}\,e^{\beta(1-\alpha)}\Big)\right)\int_\alpha^1|\varphi|^2\,dx. \tag{2.3}$$

Substituting Mu by its expression in the first term in the right-hand side of (2.3), integrating with respect to x and using the ϵ-inequalities, we have

$$\operatorname{Re}\int_0^s\int_0^1 e^{-ct}f\overline{Mu}\,dx\,dt$$

$$\leq\left(\frac{10\lambda}{\alpha^2}+\Big(32\delta_1^2\alpha^4a_1^2+16\alpha^2\Big(\frac{-\delta_1\alpha^2}{2}+\frac{\lambda}{(1-\alpha)^2a_0}\Big)^2a_1^2\Big)\,\frac{e^{2\beta(1-\alpha)}}{\lambda}\right)\int_0^s\int_0^\alpha x^2e^{-ct}|f|^2\,dx\,dt$$

$$+\left(\frac{16(1-\alpha)^6\delta^2a_1^2}{\lambda}+\frac{32}{\lambda}\,(1-\alpha)^2a_1^2\Big(\frac{|\delta|}{2}(1-\alpha)^2+\frac{\lambda}{(1-\alpha)^2a_0}\Big)^2+\frac{18\lambda}{(1-\alpha)^2}\right)\int_0^s\int_\alpha^1(x-1)^2e^{-ct}|f|^2\,dx\,dt$$

$$+\frac{1}{128}\,\frac{\lambda}{\alpha^2}\int_0^s\int_0^\alpha x^2e^{-ct}\Big|\frac{\partial u}{\partial t}\Big|^2\,dx\,dt+\frac{\lambda}{256(1-\alpha)^2}\int_0^s\int_\alpha^1(x-1)^2e^{-ct}\Big|\frac{\partial u}{\partial t}\Big|^2\,dx\,dt$$

$$+\frac{5\lambda}{32\alpha^2}\int_0^s\int_0^\alpha e^{-ct}\Big|\int_\alpha^x\frac{\partial u}{\partial t}\Big|^2\,dx\,dt+\frac{7\lambda}{32(1-\alpha)^2}\int_0^s\int_\alpha^1 e^{-ct}\Big|\int_\alpha^x\frac{\partial u}{\partial t}\Big|^2\,dx\,dt$$

$$+\frac{\lambda}{(1-\alpha)^2}\int_0^s e^{-ct}\left(\frac{1}{a(1,t)}\Big(\int_\alpha^1 f\,dx-\int_\alpha^1\frac{\partial u}{\partial t}\,dx\Big)+\frac{1}{a(0,t)}\int_0^\alpha f\,dx\right)\int_\alpha^1 ae^{\beta(1-\zeta)}\int_\alpha^\zeta\frac{\partial u}{\partial t}\,d\eta\,d\zeta\,dt. \tag{2.4}$$

Using (1.1) and (1.3), the last term in the previous inequality can be expressed as follows:

$$\frac{\lambda}{(1-\alpha)^2}\int_0^s e^{-ct}\,dt\left(\frac{1}{a(1,t)}\Big(\int_\alpha^1 f\,dx-\int_\alpha^1\frac{\partial u}{\partial t}\,dx\Big)+\frac{1}{a(0,t)}\int_0^\alpha f\,dx\right)\int_\alpha^1 ae^{\beta(1-\zeta)}\int_\alpha^\zeta\frac{\partial u}{\partial t}\,d\eta\,d\zeta\,dt$$

$$= \frac{2\lambda}{(1-\alpha)^2\alpha^2}\Big(\frac{1}{a(1,t)} - \frac{1}{a(0,t)}\Big) \int_0^s e^{-ct} \int_0^\alpha x \int_\alpha^x \frac{\partial u}{\partial t}\, dx \int_\alpha^1 a e^{\beta(1-\zeta)} \int_\alpha^\zeta \overline{\frac{\partial u}{\partial t}}\, d\eta\, dt$$

$$+ \frac{2\lambda}{(1-\alpha)^2\alpha^2} \operatorname{Re} \int_0^s e^{-ct} \Big(\frac{1}{a(0,t)} - \frac{1}{a(1,t)}\Big) \int_0^\alpha x a \frac{\partial u}{\partial x}\, dx \int_\alpha^1 a e^{\beta(1-\zeta)} \int_\alpha^\zeta \overline{\frac{\partial u}{\partial t}}\, d\eta\, dt$$

$$- \frac{\lambda}{(1-\alpha)^2\alpha^2} \operatorname{Re} \int_0^s e^{-ct} \Big(\frac{1}{a(0,t)} - \frac{1}{a(1,t)}\Big) \int_0^\alpha x^2 f\, dx \int_\alpha^1 a e^{\beta(1-\zeta)} \int_\alpha^\zeta \overline{\frac{\partial u}{\partial t}}\, d\eta\, dt.$$

From the last equality and (2.4), (2.3) becomes

$$\frac{1}{128} \frac{\lambda}{\alpha^2} \int_0^s \int_0^\alpha x^2 e^{-ct} \Big|\frac{\partial u}{\partial t}\Big|^2 dx\, dt + \frac{\lambda}{256(1-\alpha)^2} \int_0^s \int_\alpha^1 (x-1)^2 e^{-ct} \Big|\frac{\partial u}{\partial t}\Big|^2 dx\, dt$$

$$+ \Big(c - \frac{a_3}{a_0} - m_1 - 512\Big(\frac{1}{a_0} - \frac{1}{a_1}\Big)^2 \frac{\lambda a_1^4}{\alpha^3(1-\alpha)}\Big) \int_0^s \int_0^\alpha x^2 e^{-ct} \Big|\frac{\partial u}{\partial x}\Big|^2 dx\, dt$$

$$+ \Big(c - \frac{a_3}{a_0} - m_2 - 64\Big(\frac{1}{a_0} - \frac{1}{a_1}\Big)^2 a_1^4 \lambda\Big) \int_0^s \int_\alpha^1 (1-x)^2 e^{-ct} \Big|\frac{\partial u}{\partial x}\Big|^2 dx\, dt$$

$$+ \frac{\lambda}{4\alpha^2}(ca_0 - a_3) - \max\{m_3, m_4, m_5\} \int_0^s \int_0^\alpha |u|^2\, dx\, dt$$

$$+ \frac{\lambda}{4(1-\alpha)^2}(ca_0 - a_3)(-\beta a_0 - b) \int_0^s \int_\alpha^1 e^{-ct} \int_\alpha^x |u|^2\, dx\, dt$$

$$+ \int_0^s e^{-ct} \int_\alpha^1 \Big(-\delta(1-x)a(x,t)e^{\beta(1-x)} + \Big(\frac{\delta}{2}(1-x)^2 + \frac{\lambda}{(1-\alpha)^2 a(1,t)}\Big)(a_x - \beta a)e^{\beta(1-x)}\Big) \int_\alpha^x \Big|\int_\alpha^\eta \frac{\partial u}{\partial t}\, d\zeta\Big|^2 dx\, dt$$

$$+ \frac{\lambda}{32\alpha^2} \int_0^s \int_0^\alpha e^{-ct} \Big|\int_\alpha^x \frac{\partial u}{\partial t}\Big|^2 dx\, dt + \frac{1}{2048} \frac{\lambda}{(1-\alpha)^2} \int_0^s \int_\alpha^1 e^{-ct} \Big|\int_\alpha^x \frac{\partial u}{\partial t}\Big|^2 dx\, dt + \frac{\lambda a_0}{4\alpha^2} \int_0^\alpha x^2 e^{-ct} \Big|\frac{\partial u}{\partial x}\Big|^2 dx \Big|_{t=s}$$

$$+ \frac{\lambda a_0}{(1-\alpha)^2} \int_\alpha^1 (1-x)^2 e^{-ct} \Big|\frac{\partial u}{\partial x}\Big|^2 dx \Big|_{t=s} + \frac{\lambda a_0}{2\alpha^2} \int_0^\alpha e^{-ct}|u|^2\, dx \Big|_{t=s} + \frac{\lambda a_0}{(1-\alpha)^2} \int_\alpha^1 e^{-ct}|u|^2\, dx \Big|_{t=s}$$

$$\le \Big(\frac{10\lambda}{\alpha^2} + 512\Big(\frac{1}{a_0} - \frac{1}{a_1}\Big)^2 \frac{\lambda a_1^4}{\alpha^3(1-\alpha)} + 32\delta_1^2\alpha^4 a_1^2$$

$$+ 16\alpha^2\Big(\frac{-\delta_1\alpha^2}{2} + \frac{\lambda}{(1-\alpha)^2 a_0}\Big)^2 a_1^2 \frac{e^{2\beta(1-\alpha)}}{\lambda}\Big) \int_0^s \int_0^\alpha x^2 e^{-ct}|f^2\, dx\, dt$$

$$+ \Big(\frac{16(1-\alpha)^6\delta^2 a_1^2}{\lambda} + \frac{32}{\lambda}(1-\alpha)^2 a_1^2 \Big(\frac{|\delta|}{2}(1-\alpha)^2 + \frac{\lambda}{(1-\alpha)^2 a_0}\Big)^2 + \frac{18\lambda}{(1-\alpha)^2}\Big) \int_0^s \int_\alpha^1 (x-1)^2 e^{-ct}|f|^2\, dx\, dt$$

$$+ \frac{\lambda a_1}{4\alpha^2} \int_0^\alpha \frac{x^2}{2} \Big|\frac{d\varphi}{dx}\Big|^2 dx + \frac{\lambda a_1}{4(1-\alpha)^2} \int_\alpha^1 (1-x)^2 \Big|\frac{d\varphi}{dx}\Big|^2 dx$$

$$+ \frac{1}{2}\Big(\frac{\lambda}{\alpha^2} + \Big(\frac{\lambda a_0 e^\beta}{(1-\alpha)^2 a_1} + \frac{-\delta_1}{2} a_1 \alpha^2 e^{\beta(1-\alpha)}\Big)\Big) \int_0^\alpha |\varphi|^2\, dx$$

$$+\frac{1}{2}\left(\frac{\lambda}{(1-\alpha)^2}+\left(\frac{-\delta}{2}a_1(1-\alpha)^2+\frac{\lambda a_0}{(1-\alpha)^2 a_1}e^{\beta(1-\alpha)}\right)\right)\int_\alpha^1 |\varphi|^2\,dx.$$

If we take

$$c>\max\left\{m_1,m_2,\frac{4\alpha^2}{\lambda a_0}m_3,\frac{4\alpha^2}{\lambda a_0}m_4,\frac{4\alpha^2}{\lambda a_0}m_5\right\}+\frac{a_3}{a_0},$$

where

$$\begin{aligned}
m_1 &= \frac{\alpha(1-\alpha)}{\lambda}\Big[\frac{\lambda a_1^2}{(1-\alpha)^2a_0^3}+k(\alpha)\,\frac{a_1}{a_0}\Big]\max\big(|a_2|,|a_3|\big)\\
&\quad+64\,\frac{\lambda e^2\alpha^2a_1^4e^{2\beta(1-\alpha)}}{\lambda(1-\alpha)^2a_0^3}+128\,\frac{e^2a_1^4}{\alpha\lambda(1-\alpha)a_0^3}+8\,\frac{\delta_1^2\alpha^5a_1^4e^{2\beta(1-\alpha)}}{\lambda^2a_0}\\
&\quad+\frac{32\alpha(1-\alpha)^3}{\lambda^2a_0}\big[-\delta(1-\alpha)a_1^2+k(\alpha)(2b-\beta a_1)\big]^2+512\left(\frac{1}{a_0}-\frac{1}{a_1}\right)^2\frac{\lambda a_1^4}{\alpha^3(1-\alpha)},\\
m_2 &= 2\Big[\frac{a_1^2}{a_0^3}+k(\alpha)\,\frac{(1-\alpha)^2}{\lambda}\,\frac{a_1}{a_0}\Big]\max\big(|a_2|,|a_3|\big)+16\,\frac{\delta^2a_1^2(1-\alpha)^6}{\lambda^2a_0}\\
&\quad+\frac{64(1-\alpha)^6}{\lambda^2a_0}\Big[-\delta a_1^2+\frac{k(\alpha)(2b-\beta a_1)}{(1-\alpha)}\Big]^2,\\
m_3 &= \left[2b-\frac{2\delta_1\alpha^2a_1^2}{\sqrt{\lambda}}+2\alpha\Big(\frac{-\delta_1}{2}\alpha^2+\frac{\lambda a_0}{(1-\alpha)^2a_1}\Big)(2b-\beta a_0)\,\frac{e^{\beta(1-\alpha)}}{\sqrt{\lambda}}\right]^2,\\
m_4 &= \left[-\frac{8\delta(1-\alpha)^{\frac{5}{2}}a_1^2}{\sqrt{\alpha\lambda}}+\Big(-4\delta(1-\alpha)^{\frac{5}{2}}a_1+\frac{8\lambda a_1}{\sqrt{(1-\alpha)a_0}}\Big)\,\frac{(2b-\beta a_1)}{\sqrt{\alpha\lambda}}\right]^2,\\
m_5 &= \Big(\frac{1}{2}+\frac{2(1-\alpha)}{\alpha}\Big)\left[\frac{\lambda a_1^2}{(1-\alpha)^2a_0^2}+\Big(\frac{-\delta_1}{2}\alpha^2+\frac{\lambda}{(1-\alpha)^2a_0}\Big)a_1\right]\max\big(|a_2|,|a_3|\big)e^{\beta(1-\alpha)},
\end{aligned}$$

and

$$k(\alpha)=\Big(\frac{-\delta}{2}(1-\alpha)^2+\frac{\lambda}{(1-\alpha)^2a_0}\Big)$$

we deduce

$$\begin{aligned}
&\int_0^s\int_0^1\theta(x)\Big|\frac{\partial u}{\partial t}\Big|^2\,dx\,dt+\int_0^s\int_0^1\theta(x)\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dt+\int_0^1\Big(\theta(x)\Big|\frac{\partial u}{\partial x}\Big|^2+|u|^2\Big)\,dx\Big|_{t=s}\\
&\qquad\le\frac{C}{M}\,e^{cT}\Big[\int_0^T\int_0^1\theta(x)|f|^2\,dx\,dt+\int_0^1\Big(\theta(x)\Big|\frac{d\varphi}{dx}\Big|^2+|\varphi|^2\Big)\,dx\Big],
\end{aligned}\tag{2.5}$$

where

$$\begin{aligned}
C=\Bigg\{&\frac{10\lambda}{\alpha^2}+512\Big(\frac{1}{a_0}-\frac{1}{a_1}\Big)^2\frac{\lambda a_1^4}{\alpha^3(1-\alpha)}+32\delta_1^2\alpha^4a_1^2+16\alpha^2\Big(\frac{-\delta_1\alpha^2}{2}+\frac{\lambda}{(1-\alpha)^2a_0}\Big)^2a_1^2\,\frac{e^{2\beta(1-\alpha)}}{\lambda},\\
&\frac{16(1-\alpha)^6\delta^2a_1^2}{\lambda}+\frac{32}{\lambda}(1-\alpha)^2a_1^2\Big(\frac{|\delta|}{2}(1-\alpha)^2+\frac{\lambda}{(1-\alpha)^2a_0}\Big)^2+\frac{18\lambda}{(1-\alpha)^2},\\
&\frac{1}{2}\Big(\frac{\lambda a_0e^{\beta}}{(1-\alpha)^2a_1}+\frac{-\delta_1}{2}a_1\alpha^2\Big)e^{\beta(1-\alpha)},\frac{1}{2}\Big(\frac{-\delta}{2}a_1(1-\alpha)^2+\frac{\lambda a_0}{(1-\alpha)^2a_1}e^{\beta(1-\alpha)}\Big),\frac{\lambda a_1}{4\alpha^2},\frac{\lambda a_1}{4(1-\alpha)^2}\Bigg\}
\end{aligned}$$

and

$$M=\min\Bigg\{\frac{\lambda}{128},\frac{\lambda}{2561},\frac{\lambda a_0}{4},$$

$$\lambda a_0, c - \frac{a_3}{a_0} - m_1 - 512\Big(\frac{1}{a_0} - \frac{1}{a_1}\Big)^2 \frac{\lambda a_1^4}{\alpha^3(1-\alpha)}, \Big(c - \frac{a_3}{a_0} - m_2 - 64\Big(\frac{1}{a_0} - \frac{1}{a_1}\Big)^2 a_1^4\lambda\Big)\Big\} e^{-cT}.$$

Let us now make use of the fact that the choice of s is arbitrary, then (2.5) becomes

$$\int_0^s\int_0^1 \theta(x)\Big|\frac{\partial u}{\partial t}\Big|^2 dx\,dt + \int_0^s\int_0^1 \theta(x)\Big|\frac{\partial u}{\partial x}\Big|^2 dx\,dt + \sup_{0\le t\le T}\int_0^1 \Big(\theta(x)\Big|\frac{\partial u}{\partial x}\Big|^2 + |u|^2\Big)\,dx$$
$$\le \frac{C}{M}\Big[\int_0^T\int_0^1 \theta(x)|f|^2\,dx\,dt + \int_0^1 \Big(\theta(x)\Big|\frac{d\varphi}{dx}\Big|^2 + |\varphi|^2\Big)\,dx\Big]. \tag{2.6}$$

From (1.1) and (2.6) it follows that

$$\int_0^T\int_0^1 \theta(x)\Big(\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2 + \Big|\frac{\partial u}{\partial t}\Big|^2\Big)\,dx\,dt + \sup_{0\le t\le T}\int_0^1 \Big(\theta(x)\Big|\frac{\partial u}{\partial x}\Big|^2 + |u|^2\Big)\,dx$$
$$\le k\Big(\int_0^T\int_0^1 \theta(x)|f|^2\,dx + \int_0^1 \Big(\theta(x)\Big|\frac{d\varphi}{dx}\Big|^2 + |\varphi|^2\Big)\,dx\Big),$$

where

$$k^2 = \Big(\frac{((4b^2+4)+2)}{a_0^2} + \frac{C}{M}\Big).$$ □

3 Solvability of problem (1.1)–(1.4)

To prove the solvability of problem (1.1)–(1.4), it suffices to show that $R(L)$ is dense in F. The proof is based on the following

Lemma 3.1. *Suppose that the function a and its derivatives are bounded and $a(0,t) \neq a(1,t)$. Let $u \in D_0(L) = \{u \in D(L), u(x,0) = 0\}$. If for $u \in D_0(L)$ and some functions $w \in L^2(\Omega)$ we have*

$$\int_\Omega \theta(x) f\overline{\omega}\,dx\,dt = 0, \tag{3.1}$$

where

$$\theta(x) = \begin{cases} \dfrac{x^2}{\alpha^2}, & x \in (0,\alpha),\\[2mm] \dfrac{(1-x)^2}{(1-\alpha)^2}, & x \in (\alpha,1),\end{cases}$$

then w vanishes almost everywhere in Ω.

Proof. Equality (3.1) can be written as follows:

$$\int_Q \frac{\partial u}{\partial t}\,\overline{\rho}\,dx\,dt = \int_Q A(t)u\overline{\rho}\,dx\,dt, \tag{3.2}$$

where

$$\rho = \theta(x)w \quad\text{and}\quad A(t)u = \frac{\partial}{\partial x}\Big(a(x,t)\,\frac{\partial u}{\partial x}\Big).$$

We introduce the smoothing operators

$$J_\varepsilon^{-1} = \Big(I - \varepsilon\,\frac{\partial}{\partial t}\Big)^{-1} \quad\text{and}\quad (J_\varepsilon^{-1})^* = \Big(I + \varepsilon\frac{\partial}{\partial t}\Big)^{-1}$$

in the space $H^1(0,T)$ with respect to t, then these operators provide the solution of the problems

$$\begin{cases} u_\varepsilon(t) - \varepsilon \dfrac{\partial u_\varepsilon}{\partial t} = u(t), \quad u_\varepsilon(0) = 0, \\ v^*_\varepsilon(t) + \varepsilon \dfrac{\partial v^*_\varepsilon}{\partial t} = v(t), \quad v^*_\varepsilon(T) = 0. \end{cases}$$

We also have the following properties: if $g \in D(L)$, then $J_\epsilon^{-1} g \in D(L)$, and we have

$$\lim \|J_\epsilon^{-1} g - g\|_{L^2(0,T)} = 0 \text{ as } \varepsilon \to 0,$$
$$\lim \|(J_\epsilon^{-1})^* g - g\|_{L^2(0,T)} = 0 \text{ as } \varepsilon \to 0.$$

Replaicing the function u in (3.2) by the smoothing function u_ε and using the relation

$$A(t)u_\varepsilon = J_\varepsilon^{-1} A(t)u - \varepsilon J_\varepsilon^{-1} B_\epsilon(t) u_\varepsilon,$$

where

$$B_\epsilon(t)u_\varepsilon = \frac{\partial A(t)}{\partial t} u_\epsilon = \frac{\partial}{\partial x}\Big(\rho(x)\,\frac{\partial a}{\partial t}\,\frac{\partial u_\varepsilon}{\partial x}\Big),$$

we obtain

$$-\int_\Omega u \overline{\frac{\partial \rho^*_\epsilon}{\partial t}}\,dx\,dt = \int_\Omega \big(A(t)u - \epsilon B_\varepsilon(t) u_\varepsilon\big)\overline{\rho^*_\varepsilon}\,dx\,dt. \tag{3.3}$$

Since the operator $A(t)$ has a continuous inverse in $L^2(0,1)$ defined by

$$A^{-1}(t)g = \int_0^x \frac{d\zeta}{a}\Big(C_1(t) + \int_0^\zeta g(\eta)\,d\eta\Big) + \frac{1}{\alpha}\int_0^1 \frac{K(x)}{a}\Big(C_1(t) + \int_0^x g(\eta)\,d\eta\Big)\,dx,$$

where $C_1(t) = \frac{a(1,t)}{a(1,t)-a(0,t)} \int_0^1 g(\eta)\,d\eta$ and

$$K(x) = \begin{cases} x - \alpha, & (0,\alpha), \\ 0, & (\alpha,1), \end{cases}$$

then we have $\int_0^\alpha A^{-1}(t)u\,dx = 0$. Hence, the function $J_\epsilon^{-1} u = u_\varepsilon$ can be represented in the form

$$u_\varepsilon = J_\epsilon^{-1} A^{-1}(t) A(t) u,$$

and then

$$B_\varepsilon(t)g = \frac{\partial^2 a}{\partial t \partial x}\, J_\epsilon^{-1}\, \frac{C_1(t) + \int_0^\zeta g(\eta)\,d\eta}{a} + \frac{\partial a}{\partial t}\, J_\epsilon^{-1}\, \frac{g}{a} - \frac{\partial a}{\partial t}\, J_\epsilon^{-1}\, \frac{\frac{\partial a}{\partial x}}{a}\, \frac{C_1(t) + \int_0^\zeta g(\eta)\,d\eta}{a}.$$

Consequently, equality (3.3) can be written as

$$-\int_\Omega u \overline{\frac{\partial \rho^*_\epsilon}{\partial t}}\,dx\,dt = \int_\Omega A(t)u\overline{h_\epsilon}\,dx\,dt, \quad \text{where } h_\epsilon = \rho^*_\epsilon - \epsilon B^*_\epsilon(t)\rho^*_\epsilon, \tag{3.4}$$

and

$$B^*_\epsilon(t)\rho = \frac{1}{a}(J_\epsilon^{-1})^* \frac{\partial a}{\partial t}\rho^*_\varepsilon + \int_x^1 \Big(\frac{1}{a}(J_\epsilon^{-1})^* \frac{\partial^2 a}{\partial t \partial \zeta}\rho^*_\varepsilon - \frac{1}{a^2}\frac{\partial a}{\partial \zeta}(J_\epsilon^{-1})^* \frac{\partial a}{\partial t}\rho^*_\varepsilon\Big)\,d\zeta$$
$$+ \frac{a(1,t)}{a(1,t)-a(0,t)} \int_0^1 \Big(\frac{1}{a}(J_\epsilon^{-1})^* \frac{\partial^2 a}{\partial t \partial \zeta}\rho^*_\varepsilon - \frac{1}{a^2}\frac{\partial a}{\partial \zeta}(J_\epsilon^{-1})^* \frac{\partial a}{\partial t}\rho^*_\varepsilon\Big)\,dx.$$

The left-hand side of (3.4) is a continuous linear functional of u, hence the function h_ϵ has the derivatives $\frac{\partial h_\epsilon}{\partial x}$, $\frac{\partial^2 h_\epsilon}{\partial x^2} \in L^2(\Omega)$ and the following conditions are satisfied:

$$a(0,t)h_\epsilon(0,t) = a(1,t)h_\epsilon(1,t),$$
$$\frac{\partial h_\epsilon}{\partial x}(0,t) = \frac{\partial h_\epsilon}{\partial x}(1,t) = 0.$$

For a sufficiently small ϵ, the operator $I - \epsilon \frac{(J_\epsilon^{-1})^* \frac{\partial a}{\partial t}}{a}$ has a bounded inverse in $L^2(\Omega)$, so we deduce that $\frac{\partial \rho_\epsilon^*}{\partial x}$, $\frac{\partial^2 \rho_\epsilon^*}{\partial x^2} \in L^2(\Omega)$ and the conditions

$$\begin{aligned} a(0,t)\rho_\epsilon^*(0,t) &= a(1,t)\rho_\epsilon^*(1,t), \\ \frac{\partial \rho_\epsilon^*}{\partial x}(0,t) &= \frac{\partial \rho_\epsilon^*}{\partial x}(1,t) = 0 \end{aligned} \tag{3.5}$$

are satisfied. We introduce the function v such that

$$v = xw + \int_\alpha^x w\, d\zeta, \quad x \in (0,\alpha),$$
$$v = \frac{(1-x)^2}{(1-\alpha)^2}\, w, \quad x \in (\alpha,1),$$

then

$$\rho(x) = \begin{cases} \dfrac{x^2}{\alpha}\, w = \dfrac{x}{\alpha}\, v - \dfrac{1}{\alpha}\displaystyle\int_\alpha^x v\, d\zeta, & x \in (0,\alpha), \\ \dfrac{(1-x)^2}{(1-\alpha)^2}\, w = v, & x \in (\alpha,1), \end{cases}$$

and

$$\frac{\partial v}{\partial x}(0,t) = \frac{\partial v}{\partial x}(1,t) = 0 \;\text{ and }\; v(1,t) = 0.$$

From (3.5), we have

$$\frac{\partial \rho}{\partial x} = K(x)\,\frac{\partial v}{\partial x}, \quad \text{where } K(x) = \begin{cases} \dfrac{x}{\alpha}, & x \in (0,\alpha), \\ 1, & x \in (\alpha,1). \end{cases}$$

Putting

$$u = \int_0^t \exp(c\tau)\Big(\lambda_1 \int_\alpha^x av\; d\xi\, d\tau + \lambda_2 v + \frac{\lambda_1}{\alpha}\int_0^\alpha xav\; d\xi\Big)\, d\tau \tag{3.6}$$

in (3.2), integrating with respect to x and t, using (3.5) and (3.6), we obtain

$$\begin{aligned} \int_\Omega A(t)u\overline{\rho}\; dx\, dt = \frac{-1}{\lambda_2}\int_\Omega \frac{K(x)}{2}\Big(ca - \frac{\partial a}{\partial t}\Big)e^{-ct}\Big|\frac{\partial u}{\partial x}\Big|^2\; dx\, dt \\ - \frac{1}{\lambda_2}\int_0^1 \frac{K(x)}{2}\, ae^{-ct}\Big|\frac{\partial u}{\partial x}\Big|^2\; dx\Big|_{t=T} + \frac{\lambda_1}{\lambda_2}\int_\Omega K(x)a^2e^{-ct}\,\frac{\partial u}{\partial x}\,\overline{v}\; dx\, dt \end{aligned} \tag{3.7}$$

and

$$\int_0^T\int_0^\alpha \frac{\partial u}{\partial t}\,\overline{\rho}\,dx\,dt = \frac{\lambda_2}{\alpha}\int_0^T\int_0^\alpha e^{ct}x|v|^2\,dx\,dt$$
$$+\frac{\lambda_1}{\alpha}\int_0^T e^{ct}\int_0^\alpha xv\,d\zeta\overline{\int_\alpha^x av\,d\zeta}+\frac{\lambda_1}{\alpha}\int_0^T e^{ct}\int_0^\alpha xv\,d\zeta\overline{\int_0^\alpha xav\,d\zeta},$$

$$\int_0^T\int_\alpha^1 \frac{\partial u}{\partial t}\,\overline{\rho}\,dx\,dt = \lambda_2\int_0^T\int_\alpha^1 e^{ct}|v|^2\,dx\,dt$$
$$+\frac{\lambda_1}{\alpha}\int_0^T e^{ct}\int_\alpha^1 v\overline{\int_\alpha^x av\,d\zeta}\,dx\,dt+\frac{\lambda_1}{\alpha}\int_0^T e^{ct}\int_\alpha^1 v\,dx\overline{\int_0^\alpha xav\,d\zeta}\,dx\,dt.$$

Using elementary inequalities, then (3.7), becomes

$$\int_\Omega \frac{K(x)}{2}\Big(\frac{ca-\frac{\partial a}{\partial t}}{\lambda_2}-\frac{a_1^4\lambda_1^2}{\lambda_2^2}\Big)e^{-ct}\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dt$$
$$+\frac{1}{\lambda_2}\int_0^1\frac{K(x)}{2}ae^{-ct}\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\Big|_{t=T}\Big(\frac{\lambda_2}{\alpha}-\frac{7}{2}a_1\lambda_1-\frac{\alpha}{2}\Big)\int_0^T\int_0^\alpha e^{ct}x|v|^2\,dx\,dt$$
$$+\left(\lambda_2-\frac{1}{2}-\lambda_1a_1\Big(\frac{(1-\alpha)}{\alpha\sqrt{2}}-\frac{1}{2}\Big)\right)\int_0^T\int_\alpha^1 e^{ct}|v|^2\,dx\,dt\le 0,$$

we choose

$$\begin{cases}\lambda_1>0,\\ \lambda_2>\max\left(\frac{1}{2}+\lambda_1a_1\Big(\frac{(1-\alpha)}{\alpha\sqrt{2}}-\frac{1}{2}\Big),\Big(\frac{7\alpha}{2}a_1\lambda_1+\frac{\alpha^2}{2}\Big)\right),\\ c>\frac{a_1^4\lambda_1^2}{a_0\lambda_2}+\frac{a_3}{a_0}\end{cases}$$

then, we get

$$\int_Q \exp(ct)K(x)|v|^2\,dx\,dt\le 0,$$

hence $v=0$ a.e., which implies $\omega=0$. □

Theorem 3.1. *The range $R(L)$ of the operator L is dense in F.*

Proof. Since F is a Hilbert space, we have $\overline{R(L)}=F$ if and only if the relation

$$\int_Q \theta(x)f\overline{g}\,dx\,dt+\int_0^1\theta(x)\,\frac{dlu}{dx}\,\overline{\frac{d\varphi}{dx}}\,dx+\int_0^1 lu\overline{\varphi}\,dx=0. \tag{3.8}$$

for arbitrary $u\in D(L)$ and $(g,\varphi)\in F$, implies that $g=0$ and $\varphi=0$.

Putting $u\in D_0(L)$ in (3.8), we conclude from Lemma 3.1 that $g=\omega=0$, then $g=0$.

Taking $u\in D(L)$ in (3.8) yields

$$\int_0^1\theta(x)\,\frac{dlu}{dx}\,\overline{\frac{d\varphi}{dx}}\,dx+\int_0^1 lu\overline{\varphi}\,dx=0, \tag{3.9}$$

Since the two terms in the previous equality vanish independently and since the range of the trace operator L is everywhere dense in Hilbert space with the norm

$$\int_0^1 \theta(x)\Big|\frac{d\varphi}{dx}\Big|^2\,dx + \int_0^1 |\varphi|^2\,dx,$$

hence, $\varphi = 0$. Thus $\overline{R(A)} = F$. □

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(Received 20.12.2023; revised 30.05.2024; accepted 05.06.2024)

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