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GENERAL DECAY OF A SINGULAR VISCOELASTIC WAVE EQUATION WITH DISTRIBUTED DELAY AND INTEGRAL CONDITION

Abstract. In this paper, we consider a singular viscoelastic wave equation with a distributed delay and an integral condition. By introducing a suitable Lyapunov functional, under appropriate assumptions on the relaxation function and the delay weight, we establish a general decay result in which the exponential and polynomial decay are only special cases.

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1 Introduction

In the mathematical modeling of phenomena with partial differential equations (PDEs) and a set of boundary conditions, sometimes it is not possible to directly measure the boundary data such as the moment, mean, total energy or total mass, so one resorts to the integral condition

$$\int_0^l f(x)u(x,t)\,dx = g(t).$$

This kind of condition is called nonlocal, however, it has been adopted by many researchers, especially with regard to stability problems of nonlocal single systems.

The motivation for our work is based on some previous findings published in the following research papers.

In [13], Mesloub studied the solvability of the viscoelastic wave equation with frictional damping

$$u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t h(t-s)\frac{1}{x}(xu_x(s))_x\,ds + au_t = f(x,t,u,u_x), \quad (x,t) \in (0,l)\times(0,\infty),$$

with the combination of Dirichlet and integral boundary conditions, and for some properties of the relaxation function h. A similar problem, but with localized frictional damping, is considered in [3], where the authors proved the existence and general decay of a global solution. While in [18], Piskin et al. investigated the blow-up of the nonlocal singular viscoelastic system with strong damping.In the absence of the frictional dissipation, the blow-up of solutions has been proven in [19].

Moreover, in [9], the authors only care about the linear nonlocal singular viscoelastic wave equation

$$u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t h(t-s)\frac{1}{x}(xu_x(s))_x\,ds = 0, \quad (x,t) \in (0,l)\times(0,\infty),$$

under certain conditions on the function h, and prove the existence of strong solutions. But the result of general stability has been demonstrated in [2]. For a more viscoelastic problem, see [4,5,7,15–17].

In this paper, we investigate a singular viscoelastic problem with internal damping and distributed delay:

$$\begin{cases} u_{tt}(x,t) - \frac{1}{x}(xu_x(x,t))_x + \int_0^t h(t-s)\frac{1}{x}(xu_x(x,s))_x\,ds \\ \qquad + au_t(x,t) + \int_{\tau_1}^{\tau_2} b(s)u_t(x,t-s)\,ds = 0, & (x,t)\in(0,l)\times(0,\infty), \\ u(l,t) = 0, \quad \int_0^l xu(x,t)\,dx = 0, & t\in(0,\infty), \\ u(x,0) = u_1(x), \quad u_t(x,0) = u_2(x), \quad u_t(x,-t) = u_3(x,t), & x\in(0,l),\ t\in\,]0,\tau_2[\,, \end{cases} \tag{1.1}$$

where $0 < l < \infty$, h is a positive decreasing function and u_1, u_2, u_3 are given data. The term

$$\int_{\tau_1}^{\tau_2} b(s)u_t(x,t-s)\,ds$$

represents the distributed delay. Its appearance in the equation causes some disturbances.

Inspired by the previous studies, more precisely by [2], the aim of this paper is to study a general decay of solution of problem (1.1). To our knowledge, there are no works related to this issue yet.

The rest is organized as follows. In Section 2, we give some preliminaries, hypotheses and theorem on the existence and uniqueness to justify the calculations in the next section. In Section 3, we establish our general decay result.

2 Preliminary results

Let $L^2_\rho(Q)$ $(Q=(0,l)\times(0,T))$ be the Hilbert space equipped with inner product

$$(u,v)_{L^2_\rho(Q)}=\int_Q xuv\,dx\,dt$$

and associated norms

$$\|u\|_{L^2_\rho(Q)}=\int_Q x|u|^2\,dx\,dt.$$

Also, denote by $H^{1,0}_\rho(Q)$ and $H^{1,1}_\rho(Q)$ the Hilbert spaces with inner products

$$(u,v)_{H^{1,0}_\rho(Q)}=(u,v)_{L^2_\rho(Q)}+(u_x,v_x)_{L^2_\rho(Q)}$$

and

$$(u,v)_{H^{1,1}_\rho(Q)}=(u,v)_{L^2_\rho(Q)}+(u_x,v_x)_{L^2_\rho(Q)}+(u_t,v_t)_{L^2_\rho(Q)},$$

respectively.

As in [8], for the function h, we assume:

(H1) Let $h\in\mathcal{C}^1(\mathbb{R}_+,\mathbb{R}_+)$ be a decreasing function satisfying

$$h(0)>0,\qquad \int_0^\infty h(s)\,ds=\overline{h}<1.$$

(H2) There exists a nonincreasing differentiable function $\zeta:\mathbb{R}_+\to\mathbb{R}_+$ satisfying

$$h'(t)\le-\zeta(t)h(t),\quad t\ge0.$$

For the delay weight b, following [1,6], we assume:

(H3) Let $b\in L^\infty(\tau_1,\tau_2)$ and $b\ge0$ almost everywhere such that

$$\int_{\tau_1}^{\tau_1} b(s)\,ds<a,$$

which implies that there exists a positive constant c_0 such that

$$\bar{b}=a-\int_{\tau_1}^{\tau_1}\left(b(s)+\frac{c_0}{2}\right)ds\ge0. \tag{2.1}$$

Remark. Inequality (2.1) is necessary to establish the exponential stability results.

Problem (1.1) can be written in the operational form

$$\mathcal{L}u=F,$$

where $\mathcal{L}u = (L_\rho u, L_1 u, L_2 u, L_3 u)$ and $F = (0, u_1, u_2, u_3)$ with

$$\begin{aligned} L_\rho u = u_{tt}(x,t) - \frac{1}{x}\left(xu_x(x,t)\right)_x + \int_0^t h(t-s)\frac{1}{x}\left(xu_x(x,s)\right)_x ds \\ + au_t(x,t) + \int_{\tau_1}^{\tau_2} b(s)u_t(x,t-s)\,ds, \quad (x,t) \in (0,l)\times(0,\infty), \end{aligned}$$

$$u(l,t) = 0, \quad \int_0^l xu(x,t)\,dx = 0, \quad t \in (0,\infty),$$

$$L_1 u = u(x,0) = u_1(x), \quad L_2 u = u_t(x,0) = u_2(x), \quad x \in (0,l),$$
$$L_3 u = u_t(x,-t) = u_3(x), \quad x \in (0,l), \quad t \in]0,\tau_2[\,,$$

$$D(\mathcal{L}) = \left\{u \in L^2_\rho(Q), u_t, u_{tt}, u_x, u_{xx}, u_{tx} \in L^2_\rho(Q), \quad u(l,t) = 0, \quad \int_0^l xu(x,t)\,dx = 0\right\}.$$

Now, we state without proof the theorem of the existence and uniqueness, which can be proven by performing the same steps as in [9–14].

Theorem 2.1. *Let $u_1 \in H^{1,0}_\rho(0,l)$, $u_2 \in L^2_\rho(0,l)$ and $u_3 \in L^2_\rho(Q)$. Then problem (1.1) has a unique solution*

$$u \in C(0,T; H^{1,0}_\rho(0,l)) \cap C^1(0,T; L^2_\rho(0,l)) \text{ for some } T > 0.$$

We define the energy of problem (1.1) by

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^l xu_t^2\,dx + \frac{1}{2}\left(1 - \int_0^t h(s)\,ds\right)\int_0^l xu_x^2\,dx \\ + \frac{1}{2}(h \circ u_x) + \frac{1}{2}\int_0^l \int_{\tau_1}^{\tau_2} s[b(s)+c_0]\int_0^1 xu_t^2(t-ps)\,dp\,ds\,dx, \end{aligned} \tag{2.2}$$

where

$$(h \circ u_x)(t) = \int_0^l \int_0^t xh(t-s)|u_x(x,t) - u_x(x,s)|^2\,dx.$$

Lemma 2.1. *Let u be the solution of system (1.1). Then for all $t \geq 0$, we have*

$$E'(t) = \frac{1}{2}(h' \circ u_x)(t) - \frac{h(t)}{2}\int_0^l xu_x^2\,dx - \bar{b}\int_0^l xu_t^2\,dx - \frac{c_0}{2}\int_0^l \int_{\tau_1}^{\tau_2} xu_t^2(t-s)\,ds\,dx,$$

Proof. Multiplying the first equation in (1.1) by xu_t, integrating by parts over $(0,l)$ and using the same technique as in [4] for the memory term, we have

$$\begin{aligned} E'(t) = -\frac{h(t)}{2}\int_0^l xu_x^2\,dx + \frac{1}{2}(h' \circ u_x) - \int_0^l xu_t \int_{\tau_1}^{\tau_2} b(s)u_t^2(t-s)\,ds\,dx \\ - a\int_0^l xu_t^2\,dx + \int_0^l \int_{\tau_1}^{\tau_2} s[b(s)+c_0]\int_0^1 xu_{tt}u_t(t-ps)\,dp\,ds\,dx. \end{aligned} \tag{2.3}$$

It is clear that for all $t \geq 0$,

$$u_{tt}(t-ps) = -\frac{1}{s}\, u_{tp}(t-ps), \quad (p,s) \in (0,1) \times (\tau_1, \tau_2). \tag{2.4}$$

Therefore,

$$\int_0^1 u_t(t-ps) u_{tt}(t-ps)\, dp = -\frac{1}{2s} \left(u_t^2(t-s) - u_t^2(t) \right).$$

Using Cauchy–Schwarz and Young's inequalities, we get

$$\int_0^l x u_t \int_{\tau_1}^{\tau_1} b(s) u_t(t-s)\, ds\, dx \leq \frac{1}{2} \left(\int_{\tau_1}^{\tau_1} b(s)\, ds \right) \int_0^l x u_t^2\, dx + \frac{1}{2} \int_0^l \int_{\tau_1}^{\tau_1} b(s) x u_t^2(t-s)\, ds\, dx.$$

Then relationship (2.3) becomes

$$E'(t) = \frac{1}{2} (h' \circ u_x)(t) - \frac{h(t)}{2} \int_0^l x u_x^2\, dx - \bar{b} \int_0^l x u_t^2\, dx - \frac{c_0}{2} \int_0^l \int_{\tau_1}^{\tau_2} x u_t^2(t-s)\, ds\, dx,$$

where

$$\bar{b} = \left(a - \left(\int_{\tau_1}^{\tau_1} \left(b(s) + \frac{c_0}{2} \right) ds \right) \right).$$

□

Lemma 2.2 ([15], (Poincare-type inequality))**.** *Let u be a function on $H^1(0,l)$ and $u(l) = 0$. Then the following inequality holds:*

$$\int_0^l x u^2\, dx \leq 2l^2 \int_0^l x u_x^2\, dx, \quad \forall\, t \geq 0.$$

3 Asymptotic Stability

In order to prove the decay of energy, we define the Lyapunov candidate function by

$$L(t) = E(t) + \beta_1 V_1(t) + \beta_2 V_2(t) + \beta_3 V_3(t),$$

where β_1, β_2 and β_3 are positive constants, $E(t)$ is the energy given by (2.2) and

$$\begin{aligned} V_1(t) &= \int_0^l x u_t u\, dx, \\ V_2(t) &= -\int_0^l x u_t \int_0^t h(t-s)(u(t) - u(s))\, ds\, dx, \\ V_3(t) &= \int_0^L \int_{\tau_1}^{\tau_2} \int_0^1 s e^{-2ps} [b(s) + c_0] x u_t^2(t-ps)\, dp\, ds\, dx. \end{aligned} \tag{3.1}$$

Proposition. *There exist α_1 and α_2 such that*

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t), \quad \forall\, t \geq 0. \tag{3.2}$$

Proof. Using the Young and Poincaré inequalities, we obtain (3.2). □

Lemma 3.1. *The derivative of $V_1(t)$ yields*

$$V_1'(t) \le \Big(1+\frac{a^2}{\delta_1}\Big)\int_0^l xu_t^2\,dx - \Big(1-\delta_1 l^2-\overline{h}+\frac{\delta\overline{h}}{4}\Big)\int_0^l xu_x^2\,dx$$
$$+\frac{1}{\delta}\,(h\circ u_x)+\frac{1}{\delta_1}\Big(\int_{\tau_1}^{\tau_2} b(s)\,ds\Big)\int_0^l\int_{\tau_1}^{\tau_2} b(s)xu_t^2(t-s)\,ds\,dx, \tag{3.3}$$

where δ and δ_1 are positive constants.

Proof. The derivative of $V_1(t)$, the system (1.1) and the integration by parts yield

$$V_1'(t) = \int_0^l xu_t^2\,dx - \int_0^l xu_x^2\,dx + \int_0^l\int_0^t h(t-s)xu_x(s)u_x(t)\,ds\,dx$$
$$-a\int_0^l xuu_t\,dx - \int_0^l\int_{\tau_1}^{\tau_2} b(s)xuu_t(t-s)\,ds\,dx. \tag{3.4}$$

By the Young and Poincaré-type inequalities, we estimate

$$\int_0^l\int_0^t h(t-s)xu_x(s)u_x(t)\,ds\,dx \le \frac{1}{\delta}\,(h\circ u_x)+\Big(\overline{h}+\frac{\delta\overline{h}}{4}\Big)\int_0^l xu_x^2\,dx, \tag{3.5}$$

$$a\int_0^l xuu_t\,dx \le \frac{a^2}{\delta_1}\int_0^l xu_t^2\,dx + \frac{\delta_1 l^2}{2}\int_0^l xu_x^2\,dx \tag{3.6}$$

and

$$\int_0^l\int_{\tau_1}^{\tau_2} b(s)xuu_t(t-s)\,ds\,dx \le \frac{\delta_1 l^2}{2}\int_0^l xu_x^2\,dx + \frac{1}{\delta_1}\Big(\int_{\tau_1}^{\tau_2} b(s)\,ds\Big)\int_0^l\int_{\tau_1}^{\tau_1} b(s)xu_t^2(t-s)\,ds\,dx. \tag{3.7}$$

Combining (3.4)–(3.7), we obtain (3.3). ☐

Lemma 3.2. *The time derivative of $V_2(t)$ yields*

$$V_2'(t) = -\Big[\Big(\int_0^t h(s)\,ds\Big)-\frac{\delta_3\overline{h}a^2}{2}\Big]\int_0^l xu_t^2\,dx - (h'\circ u_x) + \frac{\delta_2\overline{h}}{2}\int_0^l xu_x^2$$
$$+\Big[\overline{h}+\frac{\overline{h}+1}{\delta_2}+\frac{(\overline{h}+1)l^2}{\delta_3}\Big](h\circ u_x)+\frac{\delta_3}{2}\Big(\int_{\tau_1}^{\tau_2} b(s)\,ds\Big)\int_0^l\int_{\tau_1}^{\tau_2} b(s)xu_t^2(t-s)\,ds\,dx, \tag{3.8}$$

where δ_2 and δ_3 are positive constants.

Proof. Differentiation of (3.1), system (1.1) and integration by parts give

$$V_2'(t) = -\int_0^t h(s)\,ds\int_0^l xu_t^2\,dx - (h'\circ u_x) - \int_0^l xu_x\int_0^t h(t-s)(u_x(t)-u_x(s))\,ds\,dx$$
$$+\int_0^l\int_0^t h(t-s)xu_x(s)\,ds\int_0^t h(t-s)(u_x(t)-u_x(s))\,ds\,dx$$

$$+ a\int_0^l xu_t \int_0^t h(t-s)(u(t)-u(s))\,ds\,dx$$
$$+ \int_0^l \int_{\tau_1}^{\tau_2} b(s)xu_t(t-s)\,ds \int_0^t h(t-s)(u(t)-u(s))\,ds\,dx. \tag{3.9}$$

Using the Young and Poincare-type inequalities, we obtain

$$\int_0^l xu_x \int_0^t h(t-s)(u_x(t)-u_x(s))\,ds\,dx \le \frac{1}{\delta_2}(h\circ u_x) + \frac{\delta_2\overline{h}}{4}\int_0^l xu_x^2\,dx, \tag{3.10}$$

$$\int_0^l \int_0^t h(t-s)xu_x(s)\,ds \int_0^t h(t-s)(u_x(t)-u_x(s))\,ds\,dx \le \Big(\overline{h} + \frac{\overline{h}}{\delta_2}\Big)(h\circ u_x) + \frac{\delta_2\overline{h}}{4}\int_0^l xu_x^2\,dx, \tag{3.11}$$

$$a\int_0^l xu_t \int_0^t h(t-s)x(u(t)-u(s))\,ds\,dx \le \frac{l^2}{\delta_3}(h\circ u_x) + \frac{\delta_3\overline{h}a^2}{2}\int_0^l xu_t^2\,dx \tag{3.12}$$

and

$$\int_0^l \int_{\tau_1}^{\tau_2} b(s)xu_t(t-s)\,ds \int_0^t h(t-s)(u(t)-u(s))\,ds\,dx$$
$$\le \frac{\overline{h}l^2}{\delta_3}(h\circ u_x) + \frac{\delta_3}{2}\left(\int_{\tau_1}^{\tau_2} b(s)\,ds\right)\int_0^l \int_{\tau_1}^{\tau_2} b(s)xu_t^2(t-s)\,ds\,dx. \tag{3.13}$$

Substituting (3.10)–(3.13) into (3.9), we get (3.8). □

Lemma 3.3. *The time derivative of $V_3(t)$ yields*

$$V_3'(t) = -e^{-2\tau_2}\int_0^L \int_{\tau_1}^{\tau_2} [b(s)+c_0]xu_t^2(t-s)\,ds\,dx + \int_{\tau_1}^{\tau_2}[b(s)+c_0]\,ds\int_0^L xu_t^2\,dx$$
$$-2\int_0^L \int_{\tau_1}^{\tau_2} \int_0^1 se^{-2ps}[b(s)+c_0]xu_t^2(t-ps)\,dp\,ds\,dx. \tag{3.14}$$

Proof. By deriving $V_3(t)$, using identity (2.4) and integrating by parts, we have

$$V_3'(t) = 2\int_0^L \int_{\tau_1}^{\tau_2} \int_0^1 se^{-2ps}[b(s)+c_0]xu_tu_{tt}(t-ps)\,dp\,ds\,dx$$
$$= -\int_0^L x\int_{\tau_1}^{\tau_2}[b(s)+c_0]\int_0^1 e^{-2ps}(u_t^2)_p(t-ps)\,dp\,ds\,dx$$
$$= -\int_0^L \int_{\tau_1}^{\tau_2} e^{-2s}[b(s)+c_0]xu_t^2(t-s)\,ds\,dx + \int_{\tau_1}^{\tau_2}[b(s)+c_0]\,ds\int_0^L xu_t^2\,dx$$
$$-2\int_0^L \int_{\tau_1}^{\tau_2} \int_0^1 se^{-2ps}[b(s)+c_0]xu_t^2(t-ps)\,dp\,ds\,dx.$$

Using the decay property of e^{-2s}, we obtain (3.14). □

Theorem 3.1. *Assume that **(H1)** and **(H2)** hold. Then there exist two positive constants α and C such that*

$$E(t) \le Ce^{-\alpha \int_0^t \zeta(s)\,ds}, \qquad t \ge 0. \tag{3.15}$$

Proof. Taking into account Lemmas 2.1, 3.1, 3.2 and 3.3, the derivative of $L(t)$ for all $t \ge t_0 > 0$, we obtain

$$\begin{aligned}
L'(t) = &-\Big[\overline{b} + h_0\beta_2 - \frac{\delta_3\overline{h}a^2}{2}\beta_2 - \Big(1 + \frac{a^2}{\delta_1}\Big)\beta_1 - \int_{\tau_1}^{\tau_2}[b(s)+c_0]\,ds\beta_3\Big]\int_0^l xu_t^2\,dx \\
&- \frac{1+2e^{-2\tau_2}\beta_3}{2}\,c_0\int_0^l\int_{\tau_1}^{\tau_2} xu_t^2(t-s)\,ds\,dx \\
&- \Big[e^{-2\tau_2}\beta_3 - \Big(\frac{\beta_1}{\delta_1} + \frac{\delta_3\beta_2}{2}\Big)\Big(\int_{\tau_1}^{\tau_2} b(s)\,ds\Big)\Big]\int_{\tau_1}^{\tau_2} b(s)xu_t^2(t-s)\,ds \\
&- \Big[\Big(1 - \delta_1 l^2 - \overline{h} + \frac{\delta\overline{h}}{4}\Big)\beta_1 - \frac{\delta_2\overline{h}}{2}\beta_2\Big]\int_0^l xu_x^2 \\
&- 2\beta_3\int_0^L\int_{\tau_1}^{\tau_2}\int_0^1 se^{-2ps}[b(s)+c_0]xu_t^2(t-ps)\,dp\,ds\,dx \\
&+ \Big[\frac{\beta_1}{\delta} + \Big(\overline{h} + \frac{\overline{h}+1}{\delta_2} + \frac{(\overline{h}+1)l^2}{\delta_3}\Big)\beta_2\Big](h\circ u_x) + \Big(\frac{1}{2} - \beta_2\Big)(h'\circ u_x),
\end{aligned} \tag{3.16}$$

where

$$h_0 = \int_0^{t_0} h(s)\,ds.$$

Now, it's time to set the parameters β_i and δ_i, $i = 1, 2, 3$, so that all coefficients in (3.16) are strictly negative. Then there exist two positive constants c_1 and c_2 such that

$$L'(t) \le -c_1E(t) + \frac{c_2}{2}\,(h\circ w_{xx}), \quad \forall\, t \ge t_0. \tag{3.17}$$

Multiplying (3.17) by $\zeta(t)$ and taking into account assumption **(H2)** and Lemma 2.1, we have

$$\zeta(t)L'(t) \le -c_1\zeta(t)E(t) + \frac{c_2}{2}\,\zeta(t)(h\circ w_{xx}) \le -c_1\zeta(t)E(t) - c_2E'(t), \quad \forall t \ge t_0,$$

which implies that for all $t \ge t_0$,

$$\big\{\zeta(t)L(t) + c_2E(t)\big\}' \le -c_1\zeta(t)E(t) + \zeta'(t)L(t).$$

Having $\mathcal{L}(t) = \zeta(t)L(t) + c_2E(t)$, we find that

$$d_1E(t) \le \mathcal{L}(t) \le d_1E(t), \quad d_1, d_2 > 0,$$

and exploiting the nonincreasing property of $\zeta(t)$ and (3.2), we get

$$\mathcal{L}'(t) \le -\alpha\mathcal{L}(t), \quad \forall\, t \ge t_0, \tag{3.18}$$

where α is a positive constant.

Integrating the differential inequality (3.18) over (t_0, t) and considering the fact that $\mathcal{L}(t) \sim E(t)$, we get

$$E(t) \le \frac{\mathcal{L}(t_0)}{d_1}\,e^{-\alpha\int_{t_0}^t \zeta(s)\,ds}, \quad \forall\, t \ge t_0.$$

It remains to estimate $E(t)$ on $[0,t]$. To do this, we take the decay property of the functions $E(t)$, $\mathcal{L}(t)$, $\zeta(t)$ and e^{-t}, and find that

$$E(t) \leq \begin{cases} \dfrac{\mathcal{L}(t_0)}{d_1} e^{-\alpha t_0} e^{-\alpha \int\limits_0^t \zeta(s)\,ds}, & \forall\, t \geq t_0, \\ E(0) e^{-\alpha t_0} e^{-\alpha \int\limits_0^t \zeta(s)\,ds}, & \forall\, t < t_0. \end{cases}$$

Consequently, (3.15) is established, where

$$C = \max\Big\{\frac{\mathcal{L}(t_0)}{d_1}, E(0)\Big\} e^{-\alpha t_0}\Big\}. \qquad \square$$

Example. Note that there is always a large class of relaxation functions satisfying **(H1)** and **(H2)**, our result (3.15) gives more general decay rate results. For example:

Exponential decay. Let

$$h(t) = r e^{-(1+t)^\theta}, \ \ 0 < \theta \leq 1,$$

where $r > 0$ to be chosen properly, then

$$\zeta(t) = \theta(1+t)^{\theta-1}.$$

From (3.15), we get

$$E(t) \leq C_1 e^{-d(1+t)^\theta},$$

where C_1 and d are positives constants.

Polynomial decay. Let

$$h(t) = \frac{r}{(1+t)^\eta}, \ \ \eta > 1,$$

then

$$\zeta(t) = \frac{\eta}{1+t}.$$

From (3.15), we get

$$E(t) \leq \frac{C_2}{(1+t)^{\eta\kappa}},$$

where C_2 and κ are positives constants.

Logarithmic decay. Let

$$h(t) = \frac{r}{[\ln(1+t)]^\eta}, \ \ \eta > 1,$$

then

$$\zeta(t) = \frac{\eta}{(1+t)\ln(1+t)}.$$

From (3.15), we get

$$E(t) \leq \frac{C_3}{\ln(1+t)^{\eta\kappa}},$$

where C_3 and κ are positives constants.

For more examples of other types of relaxation functions, consult [16, 17].

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