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Davit Baramidze, Salome Gabisonia, Nato Nadirashvili, Medea Tsaava

NORM CONVERGENCE FOR SOME CLASSICAL SUMMABILITY METHODS IN LEBESGUE SPACES

Abstract. In the paper, we prove norm convergence of Nörlund means and T-means in Lebesgue spaces for any $1 \leq p < \infty$.

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რეზიუმე. სტატიაში დავამტკიცეთ ნორლუნდის და T-საშუალოების ნორმით კრებადობა ლებეგის სივრცეებში ნებისმიერი $1 \leq p < \infty$-სთვის.

1 Introduction

Concerning some definitions and notations used in this introduction, we refer to Section 2. Fejér's theorem shows that (see, e.g., [1,3,4]) if one replaces ordinary summation by Fejér means σ_n defined by

$$\sigma_n f := \frac{1}{n}\sum_{k=1}^{n} S_k f,$$

then for any $1 \leq p \leq \infty$ there exists an absolute constant C_p depending only on p such that $\|\sigma_n f\|_p \leq C_p \|f\|_p$.

If we define the maximal operator σ^* of Fejér means by

$$\sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|,$$

then the weak type inequality

$$\mu(\sigma^* f > \lambda) \leq \frac{c}{\lambda}\|f\|_1 \ \ (\lambda > 0)$$

holds for any integrable function. For example, this result can be found in Zygmund [38] (see also [7, 11]) for trigonometric series, in Schipp [26] for Walsh series and in Pál, Simon [21] (see also [23,35–37]) for bounded Vilenkin series. It follows that the Fejér means with respect to trigonometric and Vilenkin systems of any integrable function converge a.e. to this function.

In this paper, we consider some more general summability methods, which are called Nörlund and T-means. In particular, the n-th Nörlund mean t_n and T-mean T_n of the Fourier series of f are defined, respectively, by

$$t_n f := \frac{1}{Q_n}\sum_{k=1}^{n} q_{n-k} S_k f, \tag{1.1}$$

$$T_n f := \frac{1}{Q_n}\sum_{k=0}^{n-1} q_k S_k f, \tag{1.2}$$

where $Q_n := \sum_{k=0}^{n-1} q_k$. Here, $\{q_k : \ k \geq 0\}$ is a sequence of nonnegative numbers, where $q_0 > 0$ and $\lim_{n\to\infty} Q_n = \infty$. Then the summability method (1.1) generated by $\{q_k : \ k \geq 0\}$ is regular if and only if (see [13])

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0.$$

Moreover, the summability method (1.2) is regular if and only if

$$\lim_{n\to\infty} Q_n = \infty.$$

It is well-known (for details, see, e.g., [25]) that every Nörlund summability method generated by the non-increasing sequence $(q_k, k \in \mathbb{N})$ is regular, but Nörlund means generated by the non-decreasing sequence $(q_k, k \in \mathbb{N})$ is not always regular. On the other hand, every T-mean generated by the non-decreasing sequence $(q_k, k \in \mathbb{N})$ is regular, but any T-mean generated by the non-increasing sequence $(q_k, k \in \mathbb{N})$ is not always regular. In this paper, we investigate only regular Nörlund and T-means.

The convergence almost everywhere (a.e.) and summability of Nörlund and T-means were studied by several authors. Here we mention the works by Bhahota, Persson and Tephnadze [5] (see also [2,4,12,24]), Tephnadze [28–32], Fridli, Manchanda, Siddiqi [6], Móricz and Siddiqi [14], Nagy [15,16] (see also [4,17–20,22,25]).

We also define the maximal operator t^* of Nörlund means by

$$t^* f := \sup_{n\in\mathbb{N}} |t_n f|.$$

If $\{q_k : k \in \mathbb{N}\}$ is non-increasing and satisfies the condition

$$\frac{1}{Q_n} = O\Big(\frac{1}{n}\Big) \text{ as } n \to \infty, \tag{1.3}$$

then the proof of the weak-type inequality

$$y\mu\big\{t^* f > y\big\} \leq c\|f\|_1, \quad f \in L^1(G_m), \quad y > 0, \tag{1.4}$$

can be found in [23]. When the sequence $\{q_k : k \in \mathbb{N}\}$ is non-decreasing, then the weak-(1,1) type inequality (1.4) holds for every maximal operator of Nörlund means. It follows that for such Nörlund means of $f \in L_1(G_m)$, we have

$$\lim_{n\to\infty} t_n f(x) = f(x) \text{ a.e. on } G_m.$$

Define the maximal operator of T-means by

$$T^* f := \sup_{n\in\mathbb{N}} |T_n f|.$$

It was proved in [33] that if $\{q_k : k \in \mathbb{N}\}$ is non-increasing, or if $\{q_k : k \in \mathbb{N}\}$ is non-decreasing and satisfies the condition

$$\frac{q_{n-1}}{Q_n} = O\Big(\frac{1}{n}\Big) \text{ as } n \to \infty, \tag{1.5}$$

then

$$y\mu\left\{T^* f > y\right\} \leq c\|f\|_1, \quad f \in L^1(G_m), \quad y > 0.$$

This implies that for such T-means and for $f \in L_1(G_m)$, we have

$$\lim_{n\to\infty} T_n f(x) = f(x) \text{ a.e. on } G_m.$$

Móricz and Siddiqi [14] investigated the approximation properties of some special Nörlund means of Walsh–Fourier series of L^p functions in a norm. In particular, they proved that if $f \in L^p(G_m)$, $1 \leq p \leq \infty$, $n = M_j + k$, $1 \leq k \leq M_j$ $(n \in \mathbb{N}_+)$ and $(q_k, k \in \mathbb{N})$ is a sequence of non-negative numbers such that

$$\frac{n^{\alpha-1}}{Q_n^\alpha}\sum_{k=0}^{n-1} q_k^\alpha = O(1) \text{ for some } 1 < \alpha \leq 2,$$

then

$$\|t_n f - f\|_p \leq \frac{C_p}{Q_n}\sum_{i=0}^{n-1} M_i q_{n-M_i}\omega_p\Big(\frac{1}{M_i}, f\Big) + C_p\omega_p\Big(\frac{1}{M_j}, f\Big),$$

when $(q_k, k \in \mathbb{N})$ is non-decreasing, while

$$\|t_n f - f\|_p \leq \frac{C_p}{Q_n}\sum_{i=0}^{n-1}(Q_{n-M_j+1} - Q_{n-M_{j+1}+1})\omega_p\Big(\frac{1}{M_i}, f\Big) + C_p\omega_p\Big(\frac{1}{M_j}, f\Big),$$

when $(q_k, k \in \mathbb{N})$ is non-increasing.

In this paper, we prove the norm convergence of Nörlund and T-means in Lebesgue spaces for some $1 \leq p < \infty$.

The paper is organized as follows. The main results are presented, proved and discussed in Section 3. In particular, Theorems 3.1 and 3.2 are the parts of this new approach. The announced results for Nörlund and T-means can be found in Theorems 4.1 and 4.2, respectively. In order not to violate the presentations in Section 3, we use Section 2 for some necessary preliminaries (e.g., definitions, notations, lemmas).

2 Preliminaries

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ denote a sequence of positive integers, not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j}'s. The direct product μ of the measures

$$\mu_k(\{j\}) := \frac{1}{m_k} \ \ (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$. In this paper, we discuss only the bounded Vilenkin groups, that is,

$$\sup_{n \in \mathbb{N}} m_n < \infty.$$

The elements of G_m are represented by the sequences $x := (x_0, x_1, \dots, x_k, \dots)$ $(x_k \in Z_{m_k})$. It is easy to provide a base for the neighborhood of G_m, namely,

$$I_0(x) := G_m,$$
$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \ \ (x \in G_m, \ \ n \in \mathbb{N}).$$

The intervals $I_n(x)$ $(n \in \mathbb{N},\, x \in G_m)$ are called Vilenkin intervals. Denote $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I}_n := G_m \setminus I_n$. Let

$$e_n := (0, \dots, 0, x_n = 1, 0, \dots) \in G_m \ \ (n \in \mathbb{N}).$$

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \ \ M_{k+1} := m_k M_k \ \ (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{k=0}^{\infty} n_j M_j, \ \ \text{where} \ \ n_j \in Z_{m_j} \ \ (j \in \mathbb{N}),$$

and only a finite number of n_j's differ from zero. Let $|n| := \max\{j \in \mathbb{N}, \ n_j \neq 0\}$. Defining $\overline{I}_n := G_m \setminus I_n$ and

$$I_N^{k,l} := \begin{cases} I_N(0, \dots, 0, x_k \neq 0, 0, \dots, 0, x_l \neq 0, x_{l+1}, \dots, x_{N-1}, \dots) & \text{for } \ 0 \leq k < l < N, \\ I_N(0, \dots, 0, x_k \neq 0, x_{k+1} = 0, \dots, x_{N-1} = 0, x_N, \dots) & \text{for } \ 0 \leq k < l = N, \end{cases}$$

we have

$$\overline{I}_N = \bigcup_{s=0}^{N-1} I_s \setminus I_{s+1} = \Big(\bigcup_{k=0}^{N-2} \bigcup_{l=k+1}^{N-1} I_N^{k,l} \Big) \bigcup \Big(\bigcup_{k=0}^{N-1} I_N^{k,N} \Big).$$

Next, we introduce on G_m an orthonormal system, which is called the Vilenkin system. First, define the complex-valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions as

$$r_k(x) := \exp\Big(\frac{2\pi \imath x_k}{m_k}\Big) \ \ (\imath^2 = -1, \ \ x \in G_m, \ \ k \in \mathbb{N}).$$

We define the Vilenkin system $\psi := (\psi_n : \ n \in \mathbb{N})$ on G_m as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \ \ (n \in \mathbb{N}).$$

Especially, we call this system the Walsh–Paley one if $m \equiv 2$ (for details, see [10, 27]). The Vilenkin system is orthonormal and complete in $L^2(G_m)$ (for details, see, e.g., [1, 27, 34]).

If $f \in L^1(G_m)$, we can define the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels with respect to the Vilenkin system ψ in the usual manner:

$$\widehat{f}(k) := \int_{G_m} f\overline{\psi}_k \, d\mu \ \ (k \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k \ \ (n \in \mathbb{N}_+, \ \ S_0 f := 0),$$

$$\sigma_n f := \frac{1}{n}\sum_{k=0}^{n-1} S_k f \ \ (n \in \mathbb{N}_+),$$

$$D_n := \sum_{k=0}^{n-1} \psi_k \ \ (n \in \mathbb{N}_+),$$

$$K_n := \frac{1}{n}\sum_{k=0}^{n-1} D_k \ \ (n \in \mathbb{N}_+).$$

Recall that (for details, see, e.g., [1, 8, 9])

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } \ x \in I_n, \\ 0 & \text{if } \ x \notin I_n, \end{cases}$$

$$n|K_n| \le c\sum_{l=0}^{|n|} M_l |K_{M_l}|$$

and

$$\int_{G_m} K_n(x)\, d\mu(x) = 1, \quad \sup_{n\in\mathbb{N}} \int_{G_m} |K_n(x)|\, d\mu(x) \le c < \infty.$$

Moreover, if $n > t$, $t, n \in \mathbb{N}$, then

$$K_{M_n}(x) = \begin{cases} \dfrac{M_t}{1 - r_t(x)}, & x \in I_t \setminus I_{t+1}, \ \ x - x_t e_t \in I_n, \\ \dfrac{M_n + 1}{2}, & x \in I_n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

3 Approximation of Vilenkin–Fejér Means

First, we prove the following important result.

Theorem 3.1. *Let $1 \le p < \infty$, $f \in L^p(G_m)$ and $n \in \mathbb{N}$. Then*

$$\|\sigma_n f - f\|_p \le c_p \omega_p\Big(\frac{1}{M_N}, f\Big) + c_p \sum_{s=0}^{N-1} \frac{M_s}{M_N}\, \omega_p\Big(\frac{1}{M_s}, f\Big).$$

Proof. Let $f \in L^p(G_m)$, $1 \le p < \infty$ and $M_N < n \le M_{N+1}$. Then

$$\begin{aligned} \|\sigma_n f - f\|_p^p &\le \|\sigma_n f - \sigma_n S_{M_N} f\|_p + \|\sigma_n S_{M_N} f - S_{M_N} f\|_p + \|S_{M_N} f - f\|_p \\ &= \|\sigma_n (S_{M_N} f - f)\|_p + \|S_{M_N} f - f\|_p + \|\sigma_n S_{M_N} f - S_{M_N} f\|_p \\ &\le c_p \omega_p\Big(\frac{1}{M_N}, f\Big) + \|\sigma_n S_{M_N} f - S_{M_N} f\|_p. \end{aligned} \tag{3.1}$$

By routine calculations, we get

$$\begin{aligned}\sigma_n S_{M_N} f - S_{M_N} f &= \frac{1}{n}\sum_{k=1}^{M_N} S_k S_{M_N} f + \frac{1}{n}\sum_{k=M_N+1}^{n} S_k S_{M_N} f - S_{M_N} f \\ &= \frac{1}{n}\sum_{k=1}^{M_N} S_k f + \frac{1}{n}\sum_{k=M_N+1}^{n} S_{M_N} f - S_{M_N} f = \frac{1}{n}\sum_{k=1}^{M_N} S_k f + \frac{n-M_N}{n} S_{M_N} f - S_{M_N} f \\ &= \frac{M_N}{n}\sigma_{M_N} f - \frac{M_N}{n} S_{M_N} f = \frac{M_N}{n}(S_{M_N}\sigma_{M_N} f - S_{M_N} f) = \frac{M_N}{n} S_{M_N}(\sigma_{M_N} f - f). \qquad (3.2)\end{aligned}$$

By using (3.2) and the fact that

$$\|S_{M_N} f\|_p \le C_p \|f\|_p, \quad f \in L_p(G_m), \quad 1 \le p < \infty,$$

we find that

$$\begin{aligned}\big\|\sigma_n S_{M_N} f - S_{M_N} f\big\|_p &= \Big(\frac{M_N}{n}\Big)^p \big\|S_{M_N}(\sigma_{M_N} f - f)\big\|_p \\ &\le \big\|S_{M_N}(\sigma_{M_N} f - f)\big\|_p \le \|\sigma_{M_N} f - f\|_p. \qquad (3.3)\end{aligned}$$

Moreover,

$$\begin{aligned}\sigma_{M_N} f(x) - f(x) &= \int_{G_m} (f(x-t) - f(x)) K_{M_N}(t)\, d\mu(t) = \int_{I_N} (f(x-t) - f(x)) K_{M_N}(t)\, d\mu(t) \\ &+ \sum_{s=0}^{N-1}\sum_{n_s=1}^{m_s-1} \int_{I_N(n_s e_s)} (f(x-t) - f(x)) K_{M_N}(t)\, d\mu(t) := I + II. \qquad (3.4)\end{aligned}$$

If we apply (2.1) and generalized Minkowski's inequality, we get

$$\|I\|_p \le \int_{I_N} \|f(x-t) - f(x)\|_p \frac{M_N - 1}{2}\, d\mu(t) \le \omega_p\Big(\frac{1}{M_N}, f\Big) \int_{I_N} \frac{M_N - 1}{2}\, d\mu(t) \le \omega_p\Big(\frac{1}{M_N}, f\Big) \qquad (3.5)$$

and

$$\begin{aligned}\|II\|_p &\le c_p M_s \sum_{s=0}^{N-1}\sum_{n_s=1}^{m_s-1} \int_{I_N(n_s e_s)} \|f(x-t) - f(x)\|_p\, d\mu(t) \\ &\le c_p M_s \sum_{s=0}^{N-1}\sum_{n_s=1}^{m_s-1} \int_{I_N(n_s e_s)} \omega_p\Big(\frac{1}{M_s}, f\Big)\, d\mu(t) \le c_p \sum_{s=0}^{N-1} \frac{M_s}{M_n} \omega_p\Big(\frac{1}{M_s}, f\Big). \qquad (3.6)\end{aligned}$$

The proof is complete by combining (3.1)–(3.6). □

Corollary 3.1. *Let $f \in \operatorname{lip}(\alpha, p)$, i.e.,*

$$\omega_p\Big(\frac{1}{M_n}, f\Big) = O\Big(\frac{1}{M_n^\alpha}\Big) \quad \text{as } n \to \infty.$$

Then

$$\|\sigma_n f - f\|_p = \begin{cases} O\Big(\dfrac{1}{M_N}\Big) & \text{if } \alpha > 1, \\ O\Big(\dfrac{N}{M_N}\Big) & \text{if } \alpha = 1, \\ O\Big(\dfrac{1}{M_N^\alpha}\Big) & \text{if } \alpha < 1. \end{cases}$$

Theorem 3.2. *Let $1 \leq p < \infty$, $f \in L^p(G_m)$ and*

$$\|\sigma_{M_n} f - f\|_p = o\Big(\frac{1}{M_n}\Big) \quad as \quad n \to \infty.$$

Then f is a constant function.

Proof. Since

$$\sigma_{M_n} f - S_{M_n} f = \frac{1}{M_n} \sum_{k=0}^{M_n - 1} k \widehat{f}(k) \psi_k,$$

by using Minkowski's integral inequality, we get

$$\Big\| \sum_{k=0}^{M_n-1} k \widehat{f}(k) \psi_k \Big\|_p \leq M_n \|\sigma_{M_n} f - f\|_p + M_n \|S_{M_n} f - f\|_p \leq 2M_n \|\sigma_{M_n} f - f\|_p \to 0 \ \text{ as } \ n \to \infty.$$

Let $0 \leq j < M_n$. Then

$$j \widehat{f}(j) = \int_{G_m} \psi_j(x) \sum_{k=0}^{M_n-1} k \widehat{f}(k) \psi_k(x) \, d\mu(x).$$

Then, using the Hölder inequality, we obtain

$$|j \widehat{f}(j)| \leq \Big(\int_{G_m} \Big| \sum_{k=0}^{M_n-1} k \widehat{f}(k) \psi_k(x) \Big|^p d\mu(x) \Big)^{1/p} \to 0 \ \text{ as } \ n \to \infty.$$

It follows that $j \widehat{f}(j) = 0$ and

$$\widehat{f}(j) = \begin{cases} \widehat{f}(0) & \text{if } \ j = 0, \\ 0 & \text{if } \ j \neq 0. \end{cases}$$

Then

$$f \sim \lim_{n \to \infty} \sum_{k=0}^{n} \Big(1 - \frac{k}{n}\Big) \widehat{f}(k) \psi_k(x) = \widehat{f}(0).$$

The proof is complete. □

4 Nörlund and T-means

From Theorem 3.1 immediately follows the following

Corollary 4.1. *Let $1 \leq p < \infty$, $f \in L^p(G_m)$ and $n \in \mathbb{N}$. Then*

$$\|\sigma_n f - f\|_p \to 0 \quad as \quad n \to \infty.$$

Based on Corollary 4.1, we can prove our next main result.

Theorem 4.1.

(a) *Let t_n be a regular Nörlund mean generated by the non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$. Then for any $f \in L^p(G_m)$, where $1 \leq p < \infty$,*

$$\lim_{n \to \infty} \|t_n f(x) - f(x)\|_p \to 0 \quad as \quad n \to \infty.$$

(b) *Let t_n be Nörlund mean generated by the non-increasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying condition (1.3). Then for any $f \in L^p(G_m)$, where $1 \leq p < \infty$,*

$$\lim_{n \to \infty} \|t_n f(x) - f(x)\|_p \to 0 \quad as \quad n \to \infty.$$

Proof. (a) Suppose that

$$\lim_{n\to\infty} \|\sigma_n f(x) - f(x)\|_p = 0.$$

If we invoke the Abel transformation, we get the following identities:

$$Q_n := \sum_{j=0}^{n-1} q_j = \sum_{j=1}^{n} q_{n-j} \cdot 1 = \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1})j + q_0 n \tag{4.1}$$

and

$$t_n f = \frac{1}{Q_n}\Big(\sum_{j=1}^{n-1} q_{n-j} - q_{n-j-1}\Big) j\sigma_j f + q_0 n\sigma_n f). \tag{4.2}$$

Combining (4.1) and (4.2), we can conclude that

$$\|t_n f(x) - f(x)\|_p \le \frac{1}{Q_n}\Big(\sum_{j=1}^{n-1}(q_{n-j} - q_{n-j-1})j\|\sigma_j f(x) - f(x)\|_p + q_0 n\|\sigma_n f(x) - f(x)\|_p\Big)$$
$$\le \frac{1}{Q_n}\sum_{j=0}^{n-1}(q_{n-j} - q_{n-j-1})j\alpha_j + \frac{q_0 n\alpha_n}{Q_n} := I + II,$$

where

$$\alpha_n := \|\sigma_n f(x) - f(x)\|_p \to 0 \;\text{ as }\; n \to \infty.$$

Since t_n are regular Nörlund means generated by the sequence of non-decreasing numbers $\{q_k : k \in \mathbb{N}\}$, we obtain

$$II \le \frac{q_0 n\alpha_n}{Q_n} \le C\alpha_n \to 0 \;\text{ as }\; n \to \infty.$$

Moreover, since α_n converges to 0, we find that there exists an absolute constant A such that $\alpha_n \le A$ for any $n \in \mathbb{N}$, and for any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $\alpha_n < \varepsilon$ when $n > N_0$. Hence

$$I = \frac{1}{Q_n}\sum_{j=1}^{N_0}(q_{n-j} - q_{n-j-1})j\alpha_j + \frac{1}{Q_n}\sum_{j=N_0+1}^{n-1}(q_{n-j} - q_{n-j-1})j\alpha_j := I_1 + I_2.$$

Since $\alpha_n \le A$, we obtain

$$I_1 = \frac{1}{Q_n}\sum_{j=1}^{N_0}(q_{n-j} - q_{n-j-1})j\alpha_j \le \frac{AN_0 q_{n-1}}{Q_n} \to 0 \;\text{ as }\; n \to \infty.$$

Moreover, by (4.1),

$$I_2 = \frac{1}{Q_n}\sum_{j=N_0+1}^{n-1}(q_{n-j} - q_{n-j-1})j\alpha_j$$
$$\le \frac{\varepsilon}{Q_n}\sum_{j=N_0+1}^{n-1}(q_{n-j} - q_{n-j-1})j \le \frac{\varepsilon}{Q_n}\sum_{j=0}^{n-1}(q_{n-j} - q_{n-j-1})j < \varepsilon.$$

We conclude that $I_2 \to 0$, as well. Thus the proof of a) is complete.

(b) In view of condition (1.3), the proof of part b) is step by step analogous to that of part (a), so, we omit the details. □

Corollary 4.2.

(a) *Let t_n be a regular Nörlund mean generated by the non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$. Then for some $f \in L^p(G_m)$, where $1 \le p < \infty$,*

$$\lim_{n\to\infty} \|t_n f(x) - f(x)\|_p \to 0 \quad as \quad n \to \infty.$$

(b) *Let t_n be Nörlund mean generated by the non-increasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying condition (1.3). Then, for some $f \in L^p(G_m)$, where $1 \le p < \infty$,*

$$\lim_{n\to\infty} \|t_n f(x) - f(x)\|_p \to 0 \quad as \quad n \to \infty.$$

Analogously, we can state the following results for T-means with respect to Vilenkin systems.

Theorem 4.2.

(a) *Let T_n be a regular T-mean generated by the non-increasing sequence $\{q_k : k \in \mathbb{N}\}$. Then for any $f \in L^p(G_m)$, where $1 \le p < \infty$,*

$$\lim_{n\to\infty} \|T_n f(x) - f(x)\|_p \to 0 \quad as \quad n \to \infty.$$

(b) *Let T_n be T-mean generated by the non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying condition (1.5). Then for any $f \in L^p(G_m)$, where $1 \le p < \infty$,*

$$\lim_{n\to\infty} \|T_n f(x) - f(x)\|_p \to 0 \quad as \quad n \to \infty.$$

Proof. The proof is step by step analogous to that of Theorem 4.1, so we omit the details. We just need to replace condition (1.3) by condition (1.5) in the proof. □

Corollary 4.3.

(a) *Let T_n be a regular T-mean generated by the non-increasing sequence $\{q_k : k \in \mathbb{N}\}$. Then for any $f \in L^p(G_m)$, where $1 \le p < \infty$,*

$$\lim_{n\to\infty} \|T_n f(x) - f(x)\|_p \to 0 \quad as \quad n \to \infty.$$

(b) *Let T_n be T-mean generated by the non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying condition (1.5). Then for any $f \in L^p(G_m)$, where $1 \le p < \infty$,*

$$\lim_{n\to\infty} \|T_n f(x) - f(x)\|_p \to 0 \quad as \quad n \to \infty.$$

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Authors' addresses:

Davit Baramidze

1. The University of Georgia, School of Science and Technology, 77a Merab Kostava Str., Tbilisi 0128, Georgia

2. Georgia and Department of Computer Science and Computational Engineering, UiT – The Arctic University of Norway, P.O. Box 385, N-8505, Narvik, Norway

E-mail: davit.baramidze@ug.edu.ge

Salome Gabisonia

The University of Georgia, School of Science and Technology, 77a Merab Kostava Str., Tbilisi 0128, Georgia

E-mail: Salo.gabisonia@gmail.co

Nato Nadirashvili

The University of Georgia, School of Science and Technology, 77a Merab Kostava Str., Tbilisi 0128, Georgia

E-mail: nato.nadirashvili@gmail.com

Medea Tsaava

The University of Georgia, School of Science and Technology, 77a Merab Kostava Str., Tbilisi 0128, Georgia

E-mail: m.tsaava@ug.edu.ge