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Anass Ourraoui

SOME EXISTENCE RESULTS FOR FOURTH-ORDER ELLIPTIC TYPE PROBLEMS WITH VARIABLE EXPONENT AND SINGULAR TERM

Abstract. The goal of this paper is to study a class of Steklov problems involving a fourth-order operator containing a singular term. Under suitable conditions, we prove the existence and multiplicity of solutions.

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1 Introduction

Exploring problems involving growth conditions dependent on $p(x)$ has attracted significant interest due to their applicability in various fields. These applications include elasticity theory, image restoration, fluids with unique properties such as thermorheological and electrorheological fluids, and even mathematical biology. For more details, we refer to [1, 10, 19].

Several researchers have investigated the existence of solutions for specific problems involving a fourth-order operator, as discussed in [2–4, 11] (particularly when the Kirchhoff function equals 1), and [7, 15, 16, 22, 24, 25, 27, 30, 34]. These studies extend beyond the classical p-biharmonic operator to cover scenarios where p is a positive constant.

Our aim is to study the following variable exponent problem:

$$(\mathcal{P}) \quad \begin{cases} \Delta^2_{m(x)} u + \omega(x) \dfrac{|u|^{l-2}u}{|x|^{2l}} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ Au = Bu + \lambda f(x,u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where:

- Ω is a bounded open domain in $\mathbb{R}^N$ $(N \geq 3)$ with a smooth boundary $\partial\Omega$,
- The operator $\Delta^2_{m(x)} u = \Delta(|\Delta u|^{m(x)-2}\Delta u)$ is the $m(x)$-biharmonic operator,
- $m \in C(\overline{\Omega})$ with $\inf_{\overline{\Omega}} m(x) > 1$ for all $x \in \overline{\Omega}$,
- $Au = |\Delta u|^{m(x)-2} \frac{\partial u}{\partial \nu}$ and $Bu = \frac{\partial u}{\partial \nu}\left(|\Delta u|^{m(x)-2}\Delta u\right)$, where ν is the outward normal vector on $\partial\Omega$,
- λ is a parameter,
- ω is a real L^∞-function with $\operatorname*{ess\,inf}_{x\in\overline{\Omega}} \omega(x) > 0$,
- l is a constant satisfying $l \in (1, N/2)$.

Throughout this paper, the function f is given by

$$f(x,u) = h(x)\big(|u|^{\alpha(x)-2}u - |u|^{\beta(x)-2}u\big),$$

where $h \in C(\overline{\Omega})$ is a nonnegative function.

We denote

$$\begin{aligned} m^- &:= \inf_{x\in\overline{\Omega}} m(x), \quad m^+ := \sup_{x\in\overline{\Omega}} m(x), \\ \alpha^- &= \inf_{\partial\Omega} \alpha(x), \quad \alpha^+ = \sup_{x\in\partial\Omega} \alpha(x), \\ \beta^- &= \inf_{\partial\Omega} \beta(x), \quad \beta^+ = \sup_{x\in\partial\Omega} \beta(x). \end{aligned}$$

The weight h satisfies

$$(H_0) \quad h \in L^{\gamma(x)}(\partial\Omega), \ \text{ with } \ \gamma \in C(\partial\Omega).$$

Now, we are ready to announce our first result.

Theorem 1.1. *Suppose that $\lambda < 0$ and $\gamma(x) > \frac{m^\partial(x)}{m^\partial(x)-1}$.*

(a) *Assume that (H_0) and (H_1) hold where*

$$(H_1) \quad 1 < l \leq m^- \leq m^+ < \alpha^- \leq \alpha^+ < \beta^- \leq \beta^+ < m^\partial_\gamma(x) = \frac{\gamma(x)-1}{\gamma(x)} m^\partial(x).$$

Then there exits $\lambda_0 < 0$ such that every $\lambda \in (\lambda_0, 0)$ is an eigenvalue of problem $(\mathcal{P})$.

(b) *Suppose* (H_0) *and* (H_2) *hold where*

$$(H_2) \quad \max\left\{2, \frac{N}{2}\right\} < \alpha^- = \alpha(x) \le \alpha^+ < m^- \le m^+ \le \beta^- \le \beta^+ < m^{\partial}_{\gamma}(x),$$

with $1 < l < \alpha^-$. *Then there exists* $A_r < 0$ *such that for each* $\lambda \in (A_r, 0)$, *problem* $(\mathcal{P})$ *has at least two weak solutions.*

Next, we suppose the following condition:

$$(H_3) \quad 1 < l < \alpha^- \le \alpha^+ < \beta^- \le \beta^+ < m(x) < \frac{N}{2} \text{ on } \overline{\Omega}.$$

The second result can be presented as follows:

Theorem 1.2. *If* $\lambda > 0$.

(c) *Under the conditions* (H_0) *with* $\gamma^- > N/2$ *and* (H_3), *there exists* $\lambda_0 > 0$ *such that for any* $\lambda \in (0, \lambda_0)$, *problem* $(\mathcal{P})$ *admits at least one solution.*

(d) *Suppose that* $\gamma(x) = \frac{m^{\partial(x)}}{m^{\partial(x)} - \alpha(x) - 1}$ *and*

$$(H_4) \quad 1 < \beta^- \le \beta(x) \le \beta^+ \le l < m^- < \frac{N}{2}, \quad \text{and } \alpha : \partial\Omega \to (-1, 0).$$

Then there exists $\lambda_* > 0$ *such that for any* $\lambda \in (0, \lambda_*)$, *problem* $(\mathcal{P})$ *admits at least one solution.*

It is worth to emphasize that the $p(x)$-biharmonic operator introduces more complex nonlinearities compared to its p-biharmonic counterpart. Typically, it is inhomogeneous and lacks a first eigenvalue, since the infimum of its principal eigenvalue is zero. Furthermore, the methods utilized in this study can be adapted to accommodate other boundary conditions, including Navier or Robin boundary value conditions.

The existence of weak solutions of the following special case of problem

$$\begin{cases} \Delta^2_{m(x)} u = \lambda |u|^{q(x)-2} u & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases}$$

has been studied in [4].

In [27], the author investigates the existence of weak solutions for a $p(x)$-biharmonic equation with the Navier boundary conditions:

$$\begin{cases} \Delta^2_{p(x)} u + |u|^{p(x)-2} u = \lambda\big(a(x)|u|^{r(x)-2}u - c(x)|u|^{p(x)-2}u\big) & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a(x)$ and $c(x)$ are nonnegative continuous functions. The study establishes the existence of at least one nontrivial weak solution.

In [26], the authors employ Ricceri's Theorem to prove the existence and multiplicity of weak solutions for a $(p(x), q(x))$-biharmonic elliptic equation involving a singular term under the Navier boundary conditions:

$$\begin{cases} \Delta^2_{p(x)} u + \Delta^2_{q(x)} u + \theta(x) \dfrac{|u|^{s-2}u}{|u|^{2s}} = \lambda f(x, u) & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega. \end{cases}$$

For the case $p = 2$ (without the Hardy singular term), the author of [29] examines the asymptotic behavior of solutions and the unique solvability of a Steklov biharmonic-type problem. Additionally, in [21], the spectral properties and positivity-preserving characteristics of the biharmonic operator are analyzed under Steklov and Navier boundary conditions, with a focus on the first Steklov eigenvalue.

In this work, we extend the existing results to the case of a $p(x)$-biharmonic equation with Steklov boundary conditions. To the best of our knowledge, this general Steklov problem involving a variable exponent has not been previously studied in the literature.

The paper is organized as follows. Section 2 reviews preliminary concepts related to variable exponent spaces, while Section 3 presents the proof of our main result and establishes the existence of solutions.

2 Preliminaries

To study problem $(\mathcal{P})$, we need some foundational theory of variable exponent Sobolev spaces. For clarity, we briefly recall the key concepts that will be used later. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$. Define

$$C_+(\overline{\Omega}) = \Big\{ m \in C(\overline{\Omega}) \mid \operatorname*{ess\,inf}_{x\in\overline{\Omega}} m(x) > 1 \Big\}.$$

For any $p(x) \in C_+(\overline{\Omega})$, let $m^- = \min_{x\in\overline{\Omega}} m(x)$ and $m^+ = \max_{x\in\overline{\Omega}} m(x)$. Additionally, we define the critical exponent

$$m_k^*(x) = \begin{cases} \dfrac{Nm(x)}{N-km(x)} & \text{if } km(x) < N, \\ +\infty & \text{if } km(x) \geq N. \end{cases}$$

Remark 2.1. It is essential to distinguish between the two types of critical exponents used in this work:

- The exponent $m_k^*(x)$, defined above, is the **Sobolev critical exponent** for the embedding of the space $W^{k,m(x)}(\Omega)$ into the Lebesgue space over the **domain** Ω. The subscript $k \in \mathbb{N}$ denotes the **order of derivatives** in the Sobolev space norm. In our analysis of the $m(x)$-biharmonic operator, the relevant exponent is $m^*(x) = m_2^*(x)$, that corresponds to the second-order derivatives in the space X, since our space X is embedded in $W^{2,m(x)}(\Omega)$.
- The exponent $m^\partial(x) = m_2^\partial(x) = \frac{(N-1)m(x)}{N-2m(x)}$, defined below, is the **trace critical exponent** for the embedding of X into the Lebesgue space on the **boundary** $\partial\Omega$. The superscript ∂ explicitly indicates its relation to the boundary. Furthermore, the notation $m^\partial_{r(x)}(x)$ (and similarly $m^\partial_\gamma(x)$) incorporates a subscript to denote a parameter $r(x)$ (or $\gamma(x)$) stemming from a **boundary weight** or from the nonlinearity in the **Steklov boundary condition**.

The variable exponent Lebesgue space is defined as

$$L^{m(x)}(\Omega) = \Big\{ v :\ \Omega \to \mathbb{R} \text{ measurable} \mid \int_\Omega |v|^{m(x)}\, dx < \infty \Big\}$$

and is equipped with the norm

$$|v|_{m(x)} = \inf \Big\{ \nu > 0 \mid \int_\Omega \Big| \frac{v}{\nu} \Big|^{m(x)} dx \leq 1 \Big\}.$$

So, $L^{m(x)}(\Omega)$ is a separable and reflexive Banach space [18].

Proposition 2.1 (cf. [16,18,33]). *Let $g(u) = \int_\Omega |v|^{m(x)}\, dx$. For $v \in L^{m(x)}(\Omega)$, the following statements hold:*

1. *If $|u|_{m(x)} \leq 1$, then $|u|^{m^+}_{m(x)} \leq g(u) \leq |u|^{m^-}_{m(x)}$.*
2. *If $|v|_{m(x)} \geq 1$, then $|v|^{m^-}_{m(x)} \leq g(u) \leq |u|^{m^+}_{m(x)}$.*

Proposition 2.2 (cf. [18]). *For any $u \in L^{m(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have*

$$\Big|\int\limits_{\Omega} uv\,dx\Big| \le \Big(\frac{1}{m^-}+\frac{1}{q^-}\Big)|u|_{m(x)}|v|_{q(x)},$$

where $\frac{1}{m(x)}+\frac{1}{q(x)}=1$.

The variable exponent Sobolev space $W^{k,m(x)}(\Omega)$ is defined as

$$W^{k,m(x)}(\Omega)=\big\{u\in L^{m(x)}(\Omega) \mid \ D^\gamma u\in L^{m(x)}(\Omega),\ |\gamma|\le k\big\},$$

where $D^\alpha u=\frac{\partial^{|\gamma|}u}{\partial x_1^{\gamma_1}\cdots\partial x_N^{\gamma_N}}$ for a multi-index $\gamma=(\gamma_1,\dots,\gamma_N)$ with $|\gamma|=\sum\limits_{i=1}^{N}\gamma_i$. The space $W^{k,m(x)}(\Omega)$, endowed with the norm $\|u\|=\sum\limits_{|\gamma|\le k}|D^\gamma u|_{p(x)}$, is a separable and reflexive Banach space (see [4,15]).

Proposition 2.3 (cf. [16,18]). *If $m,r\in C_+(\overline{\Omega})$ such that $r(x)\le m_k^*(x)$ for all $x\in\overline{\Omega}$, then there exists a continuous and compact imbedding*

$$W^{k,m(x)}(\Omega)\hookrightarrow L^{r(x)}(\Omega).$$

Proposition 2.4 (cf. [38]). *Define*

$$\rho(u)=\int\limits_{\Omega}|\Delta w|^{m(x)}\,dx+\int\limits_{\Omega}|w|^{m(x)}\,dx.$$

For $w,w_n\in W^{2,m(x)}(\Omega)$, the following properties hold:

1. *If $\|w\|\le 1$, then $\|w\|^{m^+}\le \varrho(w)\le \|w\|^{m^-}$.*
2. *If $\|w\|\ge 1$, then $\|w\|^{m^-}\le \varrho(w)\le \|w\|^{m^+}$.*
3. *$\|w_n\|\to 0$ if and only if $\rho(w_n)\to 0$.*
4. *$\|w_n\|\to +\infty$ implies $\rho(w_n)\to +\infty$.*

Lemma 2.1 (cf. [18]). *Let $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function satisfying*

$$|f(x,s)|\le a(x)+b|s|^{\frac{m_1(x)}{m_2(x)}},\ \ \forall\,(x,s)\in\overline{\Omega}\times\mathbb{R},$$

where $m_1(x),m_2(x)\in C(\overline{\Omega})$, $a(x)\in L^{m_2(x)}(\Omega)$, $m_1(x)>1$, $m_2(x)>1$, $a(x)\ge 0$, and $b\ge 0$ is a constant. Then the Nemytskii operator $N_f: L^{m_1(x)}(\Omega)\to L^{m_2(x)}(\Omega)$, defined by $N_f(u)(x)=f(x,u(x))$, is continuous and bounded.

Remark 2.2 (cf. [16]).

(1) $(W^{2,m(x)}(\Omega)\cap W_0^{1,m(x)}(\Omega),\|\cdot\|)$ is a Banach space separable and reflexive space.

(2) By the above remark and Proposition 2.2 there is a continuous and compact embedding of $W^{2,m(x)}(\Omega)\cap W_0^{1,m(x)}(\Omega)$ into $L^{s(x)}(\Omega)$, where $s(x)<m_2^*$ for all $x\in\overline{\Omega}$.

For $s\le m^-$ a.e. in Ω, it is known that

$$W_0^{1,m(x)}(\Omega)\hookrightarrow W_0^{1,s}(\Omega),\ \ W^{2,m(x)}(\Omega)\hookrightarrow W^{2,s}(\Omega).$$

In a particular case,

$$X\hookrightarrow W_0^{1,m^-}(\Omega)\cap W^{2,m^-}(\Omega).$$

In the sequel, we denote $X := (W_0^{1,m(x)}(\Omega) \cap W^{2,m(x)}(\Omega))$ equipped with the norm

$$\|u\| = \inf\left\{\nu > 0 : \left|\int\limits_{\Omega} \frac{|\Delta u|^{m(x)}}{\nu}\right| dx \leq 1\right\}$$

(see [15, 16]), and let X^* be the dual space of X.

We point out that X is also a reflexive separable space and there is a continuous and compact embedding from X into $L^{p(x)}(\Omega)$ for $1 < p(x) < m_2^*(x)$.

Let $\omega : \partial\Omega \to \mathbb{R}$ be measurable. Define the weighted variable exponent Lebesgue space by

$$L_{\omega(x)}^{m(x)}(\partial\Omega) = \left\{u : \partial\Omega \to \mathbb{R} \text{ is measurable and } \int\limits_{\partial\Omega} |\omega(x)|\, |u(x)|^{m(x)}\, d\sigma < +\infty\right\}$$

with the norm

$$|u|_{m(x),\omega(x)} = \inf\left\{\tau > 0 : \int\limits_{\partial\Omega} |\omega(x)| \left|\frac{u(x)}{\tau}\right|^{m(x)} d\sigma_x \leq 1\right\}.$$

Then $L_{\omega(x)}^{m(x)}(\partial\Omega)$ is a Banach space.

In particular, when $\omega(x) \equiv 1$ on $\partial\Omega$, $L_{\omega(x)}^{m(x)}(\partial\Omega) = L^{m(x)}(\partial\Omega)$ and $|u|_{m(x),\omega(x)} = |u|_{m(x),\partial\Omega}$.

Define

$$m^{\partial}(x) = \frac{(N-1)m(x)}{N-2m(x)}$$

and

$$m_{r(x)}^{\partial}(x) := \frac{r(x)-1}{r(x)}\, m^{\partial}(x),$$

where $x \in \partial\Omega$, $r \in C(\partial\Omega)$ with $r^-(\partial\Omega) > 1$.

Recall the following embedding theorem.

Theorem 2.1 ([13, Theorem 2.1])**.** *Assume that the boundary of Ω possesses the cone property and $m \in C(\overline{\Omega})$ with $m^- > 1$.*

Suppose that $\omega \in L^{r(x)}(\partial\Omega)$, $r \in C(\partial\Omega)$ with $r(x) > \frac{m^{\partial}(x)}{m^{\partial}(x)-1}$ for all $x \in \partial\Omega$. If $q \in C(\partial\Omega)$ and

$$1 \leq q(x) < m_{r(x)}^{\partial}(x), \ \ \forall\, x \in \partial\Omega,$$

then there exists a compact embedding $W^{k,m(x)}(\Omega) \hookrightarrow L_{\omega(x)}^{q(x)}(\partial\Omega)$.

Next, we give the classical Hardy–Rellich inequality (see [12]).

Proposition 2.5. *Let $l \in (1, N/2)$. For $u \in W^{2,l}(\Omega) \cap W_0^{1,l}(\Omega)$, we have*

$$\int\limits_{\Omega} \frac{|u(x)|^l}{|x|^{2l}}\, dx \leq \frac{1}{\mathcal{H}_l} \int\limits_{\Omega} |\Delta u|^l\, dx,$$

where

$$\mathcal{H}_l = \Big(\frac{N(l-21)(l-1)}{l^2}\Big)^l.$$

Remark 2.3. Let us denote by $\gamma'(x)$ the conjugate exponent of $\gamma(x)$ with

$$\frac{1}{\gamma'(x)} + \frac{1}{\gamma(x)} = 1.$$

In view of the assumption (H_1) or (H_2), if we put

$$q_0(x) = \frac{\gamma(x)\alpha(x)}{\gamma - \alpha(x)} \ \text{ and } \ q^0(x) = \frac{\gamma(x)\beta(x)}{\gamma - \beta(x)}\,,$$

we obtain

$$\gamma'(x)\beta(x) = \frac{\gamma(x)}{\gamma(x)-1}\,\beta(x) < m^{\partial}(x) \ \text{ and } \ \gamma'(x)\alpha(x) < m^{\partial}(x) = \frac{(N-1)m(x)}{N-2m(x)}\,.$$

Hence, by virtue of Theorem 2.1, we get $X \hookrightarrow L^{\gamma'(x)\alpha(x)}(\partial\Omega)$, $X \hookrightarrow L^{\gamma'(x)\beta(x)}(\partial\Omega)$ and also $X \hookrightarrow L^{q_0(x)}(\partial\Omega)$ (resp. $X \hookrightarrow L^{q^0(x)}(\partial\Omega)$), which are continuous and compact.

We say that $u \in X$ is a weak solution of $(\mathcal{P})$ if it satisfies

$$\int\limits_{\Omega} |\Delta u|^{m(x)-2}\Delta u \Delta v\, dx + \int\limits_{\Omega} \omega(x)\,\frac{|u|^{l-2}uv}{|x|^{2l}}\, dx - \lambda \int\limits_{\partial\Omega} h(x)\big(|u|^{\alpha(x)-2}uv - |u|^{\beta(x)-2}uv\big)\, d\sigma_x, \tag{2.1}$$

for all $v \in X$.

We shall look for a solution of $(\mathcal{P})$ by finding critical points of the functional energy $\varphi :\ X \to \mathbb{R}$ given by

$$\varphi(u) = \int\limits_{\Omega} \frac{1}{m(x)}|\Delta u|^{m(x)}\, dx + \frac{1}{l}\int\limits_{\Omega} \omega(x)\frac{|u|^{l}u}{|x|^{2l}}\, dx - \lambda \int\limits_{\partial\Omega} h(x)\Big(\frac{1}{\alpha(x)}(|u|^{\alpha(x)} - \frac{1}{\beta(x)}|u|^{\beta(x)}\Big)\, d\sigma_x, \tag{2.2}$$

where $l \in (1, N/2)$.

It is clear that $\varphi \in C^1(X, \mathbb{R})$, and for every $u \in X \setminus \{0\}$ and $v \in X$ the following holds:

$$\varphi'(u)\cdot v = \int\limits_{\Omega} |\Delta|^{m(x)-2}\Delta u \Delta v\, dx + \int\limits_{\Omega} \omega(x)\,\frac{|u|^{l-2}uv}{|x|^{2l}}\, dx - \lambda \int\limits_{\partial\Omega} h(x)\big(|u|^{\alpha(x)-2}uv - |u|^{\beta(x)-2}uv\big)\, d\sigma_x.$$

3 Proofs

Proof of Theorem 1.1.

a) We will apply the Mountain Pass Theorem. We begin by verifying the geometric conditions.

Lemma 3.1. *There exist $\lambda^* < 0$ and $\rho, r > 0$ such that for any $\lambda \in (\lambda^*, 0)$, we have*

$$\varphi(u) \geq r, \ \ \forall u \in X, \ \ \textit{with} \ \ \|u\| = \rho.$$

Proof. For $\|u\|$ sufficiently small, in view of Theorem 2.1, we have

$$\begin{aligned}\varphi(u) &= \int\limits_{\Omega} \frac{|\Delta u|^{m(x)}}{p(x)}\, dx + \int\limits_{\Omega} \frac{|\Delta u|^{l}}{l|x|^{2l}}\, dx - \lambda \int\limits_{\partial\Omega} \frac{h(x)}{\alpha(x)}\,|u|^{\alpha(x)}\, d\sigma_x + \lambda \int\limits_{\partial\Omega} \frac{h(x)}{\beta(x)}\,|u|^{\beta(x)} d\,\sigma_x \\ &\geq \frac{1}{m^+}\,\|u\|^{m^+} + \frac{1}{l}\,\|u\|^{l} - \lambda C_1\|u\|^{\alpha^+} + \lambda\|u\|^{\beta^-}.\end{aligned}$$

Since $m^+ < \alpha^+ < \beta^-$, there exist ρ, $r > 0$ and $\lambda^* < 0$ such that for any $\lambda \in (\lambda^*, 0)$, we have

$$\varphi(u) \geq r, \ \ \forall\, u \in X, \ \ \text{with} \ \ \|u\| = \rho.$$

□

Lemma 3.2. *There exist $v \in X$, $v \geq 0$ such that*

$$\lim_{t\to\infty} \varphi(tv) = -\infty.$$

Proof. Let $v \in C_0^\infty(\overline{\Omega})$, $v \neq 0$ and $t > 1$. So, we have

$$\begin{aligned}\varphi(tv) &= \Big(\int_\Omega \frac{|\Delta tv|^{m(x)}}{m(x)}\,dx\Big) - \lambda \int_{\partial\Omega} \frac{h(x)}{\alpha(x)}\,|tv|^{\alpha(x)}\,dx + \lambda \int_{\partial\Omega} \frac{h(x)}{\beta(x)}\,|tv|^{\beta(x)}\,dx \\ &\leq t^{m^+}\,\frac{1}{m^-}\int_\Omega |\Delta u|^{m(x)}\,dx + t^l \int_\Omega \frac{|\Delta u|^l}{|x|^{2l}}\,dx - \frac{\lambda t^{\alpha^-}}{\alpha^+}\int_{\partial\Omega} h|v|^{\alpha(x)}\,d\sigma_x + \lambda t^{\beta^-}\int_{\partial\Omega} h|v|^{\beta(x)}\,d\sigma_x.\end{aligned}$$

Since $\beta^- > \alpha^- > m^+$ and $\lambda < 0$, we get $\varphi(tv) \to -\infty$ as $t \to \infty$. This ends the proof of the lemma. □

Lemma 3.3. *The functional φ satisfies the Palais–Smale condition in X.*

Proof. Let $\{(u_n)\} \subset X$ be a sequence such that

$$\varphi(u_n) \to \overline{c} > 0, \quad \varphi'(u_n) \to 0 \ \text{ in } X^*, \tag{3.1}$$

where X^* is the dual space of X.

Let us show that (u_n) is bounded in X. Assume, by contradiction, the opposite. Then, passing eventually to a subsequence, still denoted by (u_n), we may assume that $\|u_n\| \to \infty$. Thus, we can consider that $\|u_n\| > 1$ for any integer n. From Proposition 2.2 we deduce that

$$\begin{aligned}\overline{c}+1 &\geq \varphi(u_n) - \frac{1}{\alpha^-}\langle \varphi' u_n, u_n\rangle \\ &\geq \Big(\frac{1}{m^+} - \frac{1}{\alpha^-}\Big)\int_\Omega |\Delta u_n|^{m(x)}\,dx + \Big(\frac{1}{l} - \frac{1}{\alpha^-}\Big)\int_\Omega |\Delta u_n|^l\,dx \\ &\qquad + \lambda\Big(\frac{1}{\alpha^-} - \frac{1}{\alpha^-}\Big)\int_\Omega h(x)|u_n|^{\alpha(x)}\,dx + \lambda\Big(\frac{1}{\beta^-} - \frac{1}{\alpha^-}\Big)\int_{\partial\Omega} h(x)|u_n|^{\beta(x)}\,dx \\ &\geq \Big(\frac{1}{m^+} - \frac{1}{\alpha^-}\Big)C\|u\|^{m^-} + C_2\Big(\frac{1}{\alpha^-} - \frac{1}{\beta^-}\Big)\|u\|^{\beta^-}.\end{aligned}$$

Since $m^+ < \alpha^- < \beta^-$, dividing the above inequality by $\|u_n\|^{m^-}$ and passing to the limit as $n \to \infty$ yields a contradiction. Thus, the sequence $(u_n)_n$ is bounded in X.

Since φ is of type S_+, we find that u_n converges strongly to u_1 in X, so we conclude that the functional φ satisfies the Palais–Smale condition. This means that all conditions of the Mountain pass Theorem are satisfied, and then $(\mathcal{P})$ has a nontrivial solution of col type. □

b) Let us recall the following interesting result.

Lemma 3.4 ([5])**.** *Let X be a real Banach space and ϕ, ψ be two continuously Gâteaux differentiable functionals such that ϕ is bounded from below and $\phi(0) = \psi(0) = 0$. Fix $r > 0$, and suppose that for each*

$$\mu \in \Big(0, \frac{r}{\sup\limits_{u\in\phi^{-1}((-\infty,r))} \psi(u)}\Big),$$

the functional $\varphi = \phi - \mu\psi$ satisfies the Palais–Smale condition, and it is unbounded from below. Then, for each

$$\mu \in \Big(0, \frac{r}{\sup\limits_{u\in\phi^{-1}((-\infty,r))} \psi(u)}\Big),$$

the functional φ admits two distinct critical points. □

If $\lambda < 0$, we set $\varphi = \phi - \lambda\psi$.

It is known that ϕ is a continuously Gâteaux differentiable functional. Further,

$$\phi'(u)\cdot v = \int_\Omega \Big(|\Delta u|^{m(x)-2}\Delta u \Delta v + \omega(x)\,\frac{|u|^{l-2}u}{|x|^{2l}} \Big)\,dx$$

for $u, v \in X$, and ψ is continuously Gâteaux differentiable (as in [2, 15]) with the compact derivative

$$\psi'(u)\cdot v = \int_{\partial\Omega} h|u|^{\alpha(x)-2}uv\,d\sigma_x - \int_{\partial\Omega} h|u|^{\beta(x)-2}uv\,d\sigma_x$$

for every $u \in X$.

It is clear that $\phi(0) = \psi(0) = 0$.

Now, let $(u_n)_n \subset X$ be a Palais–Smale sequence, which means that $\varphi(u_n)$ is bounded and $\|\varphi'(u_n)\|_{X^*} \to 0$. By the same reasoning as in the proof of Lemma 3.3, we can conclude that $(u_n)_n$ is bounded in X.

Now the sequence $(u_n)_n$ is bounded up to a subsequence denoted also by (u_n), so $u_n \rightharpoonup u$. From the compactness of ψ' we derive that $\psi'(u_n) \to \psi'(u)$. From the Palais Smale condition,

$$\varphi'(u_n) = \phi'(u_n) - \lambda\psi'(u_n) \to 0.$$

By [15, 16], ϕ' is a homeomorphism, so $u_n \to u$.

Now, we show that φ is unbounded from below for $t > 1$:

$$\begin{aligned}\varphi(tu) &= (\phi - \lambda\psi)(tu)\\ &\le \frac{1}{m^+}\big(t^{m^+}\|u\|^{m^+} + t^{m^-}\|u\|^{m^-} + t^l\|u\|^l\big) + \lambda t^{\beta^+}\int_{\partial\Omega} h|u|^{\beta(x)}\,dx - \lambda t^{\alpha^-}\int_{\partial\Omega} h|u|^{\alpha(x)}\,dx,\end{aligned}$$

since $\beta^+ > m^+ > \alpha^+ > l$, we conclude that φ is unbounded from below.

So, all hypotheses of the previous Lemma 3.4 are achieved, so for each $\lambda \in (A_r, 0)$, φ possesses at least two distinct critical points which are the weak solutions of problem $(\mathcal{P})$.

Proof of Theorem 1.2.

c) Regarding the condition (H_0) and $\gamma^- > \frac{N}{2}$, it is clear that from (H_3) we get $\gamma'(x)\alpha(x) < m^\partial(x)$ and $\gamma'(x)\beta(x) < m^{\partial(x)}$. Therefore, the embeddings $X \hookrightarrow L^{\gamma'(x)\alpha(x)}(\partial\Omega)$ and $X \hookrightarrow L^{\gamma'(x)\beta(x)}(\partial\Omega)$ are continuous and compact.

We will use the following tools.

Lemma 3.5. *There exists $v \in X$ such that $v \ge 0$, $v \not\equiv 0$ and $\varphi(tv) < 0$ for t sufficiently small.*

Proof. Let $v \in C_0^\infty(\overline{\Omega})$ such that $v \ge 0$ and $v \not\equiv 0$. For $t \in (0,1)$, we have

$$\begin{aligned}\varphi(tv) &= \int_\Omega \Big(\frac{t^{m(x)}}{m(x)}\,|\Delta(v)|^{m(x)} + \frac{t^l}{l}\,\frac{|v|^l}{|x|^{2l}}\Big)\,dx - \lambda\int_{\partial\Omega} h(x)\Big(\frac{1}{\alpha(x)}\,|tv|^{\alpha(x)} - \frac{1}{\beta(x)}\,|tv|^{\beta(x)}\Big)\,d\sigma_x\\ &\ge \frac{t^{m^-}}{m^-}\int_\Omega |\Delta v|^{m(x)}\,dx + \frac{t^l}{l}\int_\Omega \frac{|v|^l}{|x|^{2l}}\,dx + \frac{t^{\beta^-}}{\beta^-}\int_\Omega h|v|^{\beta(x)}\,dx\\ &\le C_1 t^{m^-} + C_2 t^l + C_3 t^{\beta^-} - C_4 t^{\alpha^+}.\end{aligned} \tag{3.2}$$

Hence, $\varphi(tv) < 0$ for t small enough. □

Lemma 3.6. *For $\nu \in (0,1)$, there are $\lambda_0 > 0$ and $\mu > 0$ such that $\varphi(u) \ge \mu$ for any $\lambda \in (0, \lambda_0)$ with $\|u\| = \nu$.*

Proof. From the continuous embedding from X into $L^{\gamma'(x)\alpha(x)}(\partial\Omega)$, there is $c>0$ such that

$$|u|_{\gamma'(x)\alpha(x)} \leq c\|u\|, \ \ \forall\, u \in X.$$

For $\|u\| = \nu$ small enough, we have $|u|_{\gamma'(x)\alpha(x)} < 1$. Thus,

$$\begin{aligned}\varphi(u) \geq \frac{1}{m^+}\,\|u\|^{m^+} - \frac{1}{\alpha^-}\,\lambda \int_{\partial\Omega} |u|^{\alpha(x)}\,dx &\geq \frac{1}{m^+}\,\|u\|^{m^+} - \frac{2}{\alpha^-}\,\lambda |u|_{\gamma(x)} \big||u|^{\alpha(x)}\big|_{\gamma'(x)} \\ &\geq \frac{1}{m^+}\,\|u\|^{m^+} - \frac{2}{\alpha^-}\,|h|_{\gamma(x)}\|u\|^{\alpha^-} = \nu^{\alpha^-}\Big(\frac{1}{m^+}\,\nu^{m^+-\alpha^-} - \frac{2}{\alpha^-}\,C_0|h|_{\gamma(x)}\Big). \quad (3.3)\end{aligned}$$

So, $\varphi(u) \geq \mu$, where

$$\lambda_0 = \frac{\nu^{m^+-\alpha^-}}{2\nu^+ \frac{\alpha^-}{2C_0|h|_{\gamma(x)}}} \ \text{ and } \ \mu = \frac{\nu^{m^+}}{2m^+}\,. \qquad \square$$

From Lemma 3.6, there exits $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$, we have

$$\inf_{\partial B_\nu(0)} \varphi > 0, \ \text{ where } \ \partial B_\nu = \big\{u \in B_\nu(0) \mid \ \|u\| = \nu\big\}. \quad (3.4)$$

For $u \in B_\nu(0)$, we assert that

$$\varphi(u) \geq \frac{1}{m^+}\,\|u\|^{m^+} - \frac{2}{\alpha^-}\,\lambda C_0 |h|_{\gamma(x)} \|u\|^{\gamma^-}. \quad (3.5)$$

In view of (3.4) and (3.5), this yields

$$-\infty < c^* = \inf_{\overline{B_\nu}}(0)\varphi < 0.$$

Let $c_* = \min_{\overline{B}_\nu} \varphi(u)$. Through a simple calculation, we obtain $c_* < 0$. Applying Ekeland's variational principle as stated in [14], there exists a sequence $(u_n)_n$ satisfying

$$c_* \leq \varphi(u_n) \leq c_* + \frac{1}{n} \quad (3.6)$$

and

$$\varphi(v) \leq \varphi(u_n) - \frac{1}{n}\,\|u_n - v\|, \ \ \forall\, v \in \overline{B}_\nu.$$

Consequently, $\|u_n\| \leq \nu$ for sufficiently large $n > 1$.

The case $\|u_n\| = \nu$ is impossible. Indeed, from Lemma 3.6, we have $\varphi(u_n) \geq \mu > 0$. Taking the limit as $n \to \infty$ and combining this with (3.6), we arrive at

$$0 < \mu \leq c_* < 0,$$

which is a contradiction. Therefore, $\|u_n\| < \nu$.

Next, we verify that $\varphi'(u_n) \to 0$ in X. Let $u \in X$ with $\|u\| = 1$, and define $\omega_n = u_n + \tau u$. For a fixed $n > 1$, we have

$$\|\omega_n\| \leq \|u_n\| + \tau < \mu$$

for sufficiently small $\tau > 0$. This implies that

$$\varphi(u_n + \tau u) \geq \varphi(u_n) - \frac{\tau}{n}\,\|u\|.$$

Dividing by τ and taking the limit as $\tau \to 0$, we obtain

$$\varphi'(u_n)\cdot u \geq -\frac{1}{n}\,.$$

Since $\|u\| = 1$, it follows that

$$|\varphi'(u_n) \cdot u| \leq \frac{1}{n},$$

which shows $\varphi'(u_n) \to 0$ in X^*. By standard arguments, there exists $u_* \in X$ such that $\varphi'(u_*) = 0$ and $\varphi(u_*) = c_* < 0$.

d) Let $\lambda > 0$ and $\alpha \in (-1, 0)$. For simplicity, consider $\delta = -\alpha$, which means $\delta \in (0, 1)$. Then problem $(\mathcal{P})$ will take the form:

$$\begin{cases} \Delta^2_{m(x)} u + \omega(x) \dfrac{|u|^{l-2} u}{|x|^{2l}} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\Delta u|^{m(x)-2} \dfrac{\partial u}{\partial \nu} = \dfrac{\partial u}{\partial \nu} \left(|\Delta u|^{m(x)-2} \Delta u \right) + \lambda h(x) \Big(\dfrac{1}{|u|\delta}\, u - |u|^{\beta(x)-2} u \Big) = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus the energy functional is still defined in X as

$$\varphi(u) = \int\limits_{\Omega} \frac{1}{m(x)} |\Delta|^{m(x)}\, dx + \frac{1}{l} \int\limits_{\Omega} \omega(x) \frac{|u|^l u}{|x|^{2l}} - \lambda \int\limits_{\partial\Omega} h(x) \Big(\frac{1}{-\delta(x) + 1} |u|^{1-\delta(x)} - \frac{1}{\beta(x)} |u|^{\beta(x)} \Big)\, d\sigma_x.$$

Similarly as in [36], we can verify that the energy functional φ is weakly lower semi-continuous. From the Hölder inequality, we have

$$\int\limits_{\partial\Omega} h|u|^{-\sigma(x)+1} \leq |h|_{\frac{m^{\partial(x)}}{m^{\partial(x)}} + (\sigma(x)-1)} |u|_{m^{\partial(x)}(-\sigma(x)+1)}. \tag{3.7}$$

Moreover, let $\| u \| > 1$. From Proposition 2.4 and (3.7), it follows that

$$\varphi(u) \geq \frac{1}{m^+} \|u\|^{m^-} - C_1 \|u\|^{-\delta^+ + 1},$$

in view of (H_1), $\delta^- + 1 < m^-$, so φ is coercive and has a minimizer, which is a solution of $(\mathcal{P})$. This minimizer is nonzero. In fact, for $t > 0$ small enough and $v_1 \in X$, $v_1 \not\equiv 0$, we have

$$\begin{aligned} \varphi(tv_1) &= \Big(\int\limits_{\Omega} \frac{1}{m(x)} |\Delta t v_1^{m(x)}\, dx \Big) + \int\limits_{\Omega} \frac{\omega(x)}{l} \frac{|u|^l}{|x|^{2l}} - \lambda \int\limits_{\partial\Omega} \frac{1}{-\delta(x)+1} h(x) |tv_1|^{-\delta(x)+1}\, d\sigma_x \\ &\leq +\lambda \int\limits_{\partial\Omega} \frac{1}{\beta(x)} h(x) |tv_1|^{\beta(x)}\, d\sigma_x \\ &\leq \Big(\int\limits_{\Omega} \frac{t^{m(x)}}{m(x)} |v_1|^{m(x)}\, dx \Big) + t^l \int\limits_{\Omega} \frac{\omega(x)}{l} \frac{|v_1|^l}{|x|^{2l}} + C_2 t^{\beta^-} \int\limits_{\partial\Omega} |v|_1^{\beta(x)}\, d\sigma_x - C_3 t^{1-\delta^-} \int\limits_{\partial\Omega} |v|_1^{1-\delta^-}\, d\sigma_x, \\ &< 0, \end{aligned}$$

since $1 - \delta^- < 1$.

As is known, X is a reflexive Banach space, and since φ is coercive and weakly lower semi-continuous, $(\mathcal{P})$ admits a nontrivial weak solution of a global minimum type. □

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Authors' addresses:

Anass Ourraoui
Department of Mathematics, Faculty of Sciences, University Mohamed I, Oujda, Morocco
E-mail: a.ourraoui@gmail.com