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TIME-VARYING LYAPUNOV FUNCTIONS FOR NONAUTONOMOUS CONFORMABLE FRACTIONAL-ORDER SYSTEMS

Abstract. In this paper, we present a new inequality which involves the conformable fractional derivative of the product of two continuously differentiable functions, and establish its various properties. The inequality and its properties enable us to construct potential time-varying Lyapunov functions for the stability and stabilization of conformable fractional order systems. Some examples are given to illustrate the efficiency of the obtained results.

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1 Introduction

The history of fractional systems is more than three centuries old, yet it received much attention and interest only in the past 20 years; the reader may refer to [7, 11] for the theory and applications of fractional calculus. The idea of fractional calculus was mentioned in 1695 by Leibniz and L'Hospital. The first fractional order calculus was defined by Riemann and Liouville in the nineteenth century. However, this fractional calculus started to be interesting for engineers only in the late 1960s, especially when it was observed that the description of some systems is more accurate when the fractional derivative is used [10]. Several definitions of the fractional derivative exist in the literature such as Riemann–Liouville, Caputo, Riesz, Riesz–Caputo, Grunwald–Letnikov and WeyI. All these fractional derivatives are defined via fractional integrals, so they represent nonlocal properties.

It is known that the above fractional derivatives in general do not follow chain rule, product rule, quotient rule, etc. Due to these discrepancies, mathematical analysis becomes difficult. Lately, a new well-behaved definition of the fractional derivative, known as the conformable fractional derivative, was introduced in [4]. In 2015, Thabet Abdeljawad (see [1]) proceeded on to develop the definition. Some basic concepts about conformable fractional derivative such as chain rule, Grunwall's inequality, exponential functions and Lyapunov inequality were studied in [3, 9, 13].

On the other hand, the stability of the differential system is also attracted for researchers, that is because the stable system is very important in our life. Recently, stability problems of nonlinear fractional systems have been extensively investigated by many authors [8, 12, 15]. In nonlinear systems, only the Lyapunov direct method (there exists a positive definite scalar function such that the derivative of this function decreases along the orbits of the system) provides a way to analyze the stability of a system without explicitly solving the differential equation. This method generalizes the idea which shows that the system is asymptotically stable if there exists some Lyapunov function for the system. The fractional Lyapunov direct method has a limitation and a difficulty, when it comes to the stability analysis of fractional order systems. In [14], Abdourazek Souahi et al. studied the stability of conformable fractional order nonlinear systems by using the Lyapunov function. In [2], the authors used the Lyapunov–Razumikhin theorem for the uniform stability and uniform asymptotic stability of the conformable fractional system with a delay.

It may be noted that all these works have been focused on time-invariant or autonomous Lyapunov functions (functions depending only on the state variables of a fractional-order system) which are continuously differentiable. However, in general, it is very essential to construct continuously differentiable time-varying or nonautonomous Lyapunov functions (functions in terms of independent variable 'time' and the dependent variables 'state variables') for the stability of nonautonomous conformable fractional order systems.

The purpose of this paper to construct the continuously differentiable time-varying Lyapunov functions for the stability analysis of nonautonomous conformable fractional-order systems. First, we propose an inequality which involves the conformable fractional derivative of the product of two functions. Further, we show that it is possible to ensure the stability and stabilization of nonautonomous fractional-order systems based on the time-varying Lyapunov functions. Finally, we illustrate the results by numerical examples.

2 Notation and preliminaries

We start by recalling some classical notations and definitions that will be useful throughout the paper.

$\mathbb{R}^+$ denotes the set of all real nonnegative numbers

$\mathbb{R}^n$ denotes the n-dimensional space;

$\langle x, y\rangle$ or $x^T y$ denotes the scalar inner product of two vectors $x, y \in \mathbb{R}^n$;

$\|x\|$ denotes the Euclidean vector norm of x;

$\mathbb{R}^{n\times m}$ is the set of all $n \times m$ matrices;

I_n denotes the identity matrix;

A^T denotes the transpose matrix of A; A is symmetric if $A^T = A$;

The matrix $A \in \mathbb{R}^{n\times n}$ is bounded on $\mathbb{R}^+$ if

$$\exists\, M > 0 \ \text{ such that } \ \sup_{t\geq 0} \|A(t)\| \leq M;$$

The matrix function $A(t)$ is positive definite ($A(t) > 0$) if there exists a constant $c > 0$ such that

$$\langle A(t)x, x\rangle \geq c\|x\|^2 \ \text{ for all } \ x \in \mathbb{R}^n, \ \ t \geq 0.$$

Definition 2.1. Given a function h defined on $[a, \infty)$, then the conformable fractional derivative starting from a of a function h of order α is defined by

$$T_a^\alpha h(t) = \lim_{\varepsilon\to 0} \frac{h(t + \varepsilon(t-a)^{1-\alpha}) - h(t)}{\varepsilon}$$

for all $t > a$, $\alpha \in (0, 1]$. If $T_a^\alpha h(t)$ exists for all $t \in (a, b)$, for some $b > a$, and $\lim\limits_{t\to a^+} T_a^\alpha h(t)$ exists, then by definition,

$$T_a^\alpha h(a) = \lim_{t\to a^+} T_a^\alpha h(t).$$

In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is α-differentiable.

Theorem 2.1 ([4])**.** If a function $h : [0, \infty) \to \mathbb{R}$ is α-differentiable at $t_0 > 0$, $\alpha \in (0, 1]$, then h is continuous at t_0.

Definition 2.2 ([1, 4])**.** The conformable fractional integral starting from a of a function h of order $0 < \alpha \leq 1$ is defined by

$$I_a^\alpha h(t) = \int_a^t (x-a)^{\alpha-1} h(x)\, dx.$$

Lemma 2.1 ([1, 4])**.** Assume that h is a continuous function on (a, ∞) and $0 < \alpha < 1$. Then for all $t > a$, we have

$$T_a^\alpha I_a^\alpha h(t) = h(t).$$

Lemma 2.2 ([1,4])**.** Let $h : [a, \infty) \to \mathbb{R}$ be a continuous function such that $T_a^\alpha h(t)$ exists on (a, ∞). If $T_a^\alpha h(t) \geq 0$ (respectively $T_a^\alpha h(t) \leq 0$), for all $t \in (a, \infty)$, then the graph of h is increasing (respectively decreasing).

Lemma 2.3 ([14])**.** Let $h : [a, \infty) \to \mathbb{R}$ such that $T_a^\alpha h(t)$ exists on (a, ∞). Then $T_a^\alpha h^2(t)$ exists on (a, ∞) and

$$T_a^\alpha h^2(t) = 2h(t) T_a^\alpha h(t), \ \ \forall\, t > a.$$

Lemma 2.4. Let $x(t) \in \mathbb{R}$ be a real continuously differentiable function. Then for any $p = 2^n, n \in \mathbb{N}$, we have

$$T_a^\alpha x^p(t) = p x^{(p-1)}(t) T_a^\alpha x(t), \tag{2.1}$$

where $0 < \alpha < 1$ is the fractional order.

Proof. The case $p = 2$ is established firstly (Lemma 2.3). So, by the use of recursion, for any $p = 2^n$, $n \in \mathbb{N}$, we have

$$T_a^\alpha x^p(t) = 2x^{p/2}(t) T_a^\alpha x^{p/2}(t) = 2^2 x^{(p/2+p/2^2)}(t) T_a^\alpha x^{p/2^2}(t) = \cdots = p x^{(p-1)}(t) T_a^\alpha x(t).$$

Thus inequality (2.1) is proved completely. $\square$

Remark 2.1. Let $h : [a, \infty) \to \mathbb{R}^n$ be such that $T_a^\alpha h(t)$ exists on (a, ∞). Then $T_a^\alpha h^T h(t)$ exists on (a, ∞) and

$$T_a^\alpha h^T h(t) = 2h(t)^T T_a^\alpha h(t), \ \ \forall\, t > a,$$

where h^T is the transpose of the vector h.

Let us consider the system of fractional differential equations with the conformable derivative

$$T_{t_0}^\alpha x = f(t, x), \ \ t > t_0, \ \ x(t_0) = x_0, \tag{2.2}$$

where $\alpha \in (0, 1]$ and $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a given nonlinear function satisfying $f(t, 0) = 0$, $\forall\, t \in \mathbb{R}^+$. Suppose that the function f is smooth enough to guarantee the existence of a global solution $x(t) = x(t, t_0, x_0)$ of system (2.2) for each initial condition (t_0, x_0).

Definition 2.3. The zero solution of system (2.2) is said to be:

(i) stable if

$$\forall\, \varepsilon > 0, \ \ \forall\, t_0 \geq 0, \ \ \delta = \delta(t_0, \varepsilon) > 0$$

such that

$$\|x_0\| < \delta \Longrightarrow \|x(t)\| < \varepsilon, \ \ \forall\, t \geq t_0;$$

(ii) attractive if there exists a neighborhood $\mathcal{V}$ of 0 such that for any initial condition x_0 belonging to $\mathcal{V}$, the corresponding solution $x(t, t_0)$ is defined for all $t \geq t_0$ and $\lim\limits_{t\to+\infty} x(t) = 0$. If $\mathcal{V} = \mathbb{R}^n$, $x = 0$ is globally attractive;

(iii) asymptotically stable if it is stable and attractive;

(iv) globally asymptotically stable (GAS) if it is stable and globally attractive.

Definition 2.4 ([4])**.** A scalar continuous function $\alpha(r)$ defined for $r \in [0, a[$ is said to belong to the class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

It is said to belong to the class $\mathcal{K}_\infty$ if it is defined for all $r \geq 0$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Theorem 2.2 ([14])**.** Let $x = 0$ be an equilibrium point for system (2.2). Assume that there exist a continuous function $V(t, x)$ and the class $\mathcal{K}$ functions α_i $(i = 1, 2, 3)$ satisfying

(A1) $\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$;

(A2) $V(t, x(t))$ has a fractional derivative of order α for all $t > t_0 \geq 0$;

(A3) $T_{t_0}^\alpha V(t, x(t)) \leq -\alpha_3(\|x\|)$.

Then the origin of system (2.2) is asymptotically stable. Moreover, if $\alpha_i \in \mathcal{K}_\infty$ $(i = 1, 2, 3)$, then the origin of system (2.2) is globally asymptotically stable.

Definition 2.5 ([1])**.** The fractional conformable exponential function is defined for every $s \geq 0$ by

$$E_\alpha(\lambda, s) = \exp\Big(\lambda\, \frac{s^\alpha}{\alpha}\Big),$$

where $\alpha \in (0, 1)$ and $\lambda \in \mathbb{R}$.

Definition 2.6 (Fractional Exponential Stability [1])**.** The origin of system (2.2) is said to be fractional exponentially stable if

$$\|x(t)\| \leq K\|x_0\| E_\alpha(-\lambda, t - t_0), \ \ t \geq t_0,$$

where $\lambda, K > 0$.

Lemma 2.5 ([14])**.** Let $0 < \alpha < 1$ and $g : [t_0, \infty) \to \mathbb{R}_+$ be a continuous function and α-differentiable on (t_0, ∞) such that

$$T_{t_0}^\alpha g(t) \leq -\lambda g(t),$$

where λ is a positive constant. Then

$$g(t) \leq E_\alpha(-\lambda, t - t_0) g(t_0).$$

3 Main result

3.1 Inequalities

Lemma 3.1. Let $\varphi : [t_0, \infty) \to \mathbb{R}$ be monotonically decreasing and α-differentiable at a point $t > t_0$. Suppose $x : [t_0, \infty) \to \mathbb{R}$ is a positive and continuously differentiable function. Then the inequality

$$T^{\alpha}_{t_0}(\varphi x)(t) \leq \varphi(t) T^{\alpha}_{t_0} x(t), \ \ \forall\, t \geq t_0, \ \ \forall\, \alpha \in (0, 1], \tag{3.1}$$

holds.

Proof. Let

$$F_\alpha(t) = T^{\alpha}_{t_0}(\varphi x)(t) - \varphi(t) T^{\alpha}_{t_0} x(t).$$

In order to prove that inequality (3.1) holds, it is sufficient to check that $F_\alpha(t) \leq 0$, $\forall\, t > t_0$, $\forall\, \alpha \in (0, 1)$. Inequality (3.1) holds for the case $\alpha = 1$, and to the case $t = t_0$ for $\alpha \in (0, 1)$. Note that by the definition of conformable fractional derivative, we have

$$\begin{aligned}
T^{\alpha}_{t_0}(\varphi x)(t) &= \lim_{\varepsilon \to 0} \frac{\varphi(t + \varepsilon(t - t_0)^{1-\alpha}) x(t + \varepsilon(t - t_0)^{1-\alpha}) - \varphi(t) x(t)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\varphi(t + \varepsilon(t - t_0)^{1-\alpha}) x(t + \varepsilon(t - t_0)^{1-\alpha}) - x(t) \varphi(t + \varepsilon(t - t_0)^{1-\alpha})}{\varepsilon} \\
&\qquad + \lim_{\varepsilon \to 0} \frac{x(t) \varphi(t + \varepsilon(t - t_0)^{1-\alpha}) - \varphi(t) x(t)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{x(t + \varepsilon(t - t_0)^{1-\alpha}) - x(t)}{\varepsilon}\, \varphi(t + \varepsilon(t - t_0)^{1-\alpha}) \\
&\qquad + x(t) \lim_{\varepsilon \to 0} \frac{\varphi(t + \varepsilon(t - t_0)^{1-\alpha}) - \varphi(t)}{\varepsilon} \\
&= T^{\alpha}_{t_0}(x)(t) \lim_{\varepsilon \to 0} \varphi(t + \varepsilon(t - t_0)^{1-\alpha}) + x(t) T^{\alpha}_{t_0}(\varphi)(t)
\end{aligned}$$

So,

$$\begin{aligned}
F_\alpha(t) &= T^{\alpha}_{t_0}(\varphi x)(t) - \varphi(t) T^{\alpha}_{t_0} x(t) \\
&= T^{\alpha}_{t_0}(x)(t) \lim_{\varepsilon \to 0} \varphi\big(t + \varepsilon(t - t_0)^{1-\alpha}\big) + x(t) T^{\alpha}_{t_0}(\varphi)(t) - \varphi(t) T^{\alpha}_{t_0} x(t) \\
&= T^{\alpha}_{t_0}(x)(t) \Big(\lim_{\varepsilon \to 0} \varphi\big(t + \varepsilon(t - t_0)^{1-\alpha}\big) - \varphi(t) \Big) + x(t) T^{\alpha}_{t_0}(\varphi)(t)
\end{aligned}$$

Since φ is continuous at t, we have

$$\lim_{\varepsilon \to 0} \varphi\big(t + \varepsilon(t - t_0)^{1-\alpha}\big) - \varphi(t) = 0$$

Using Lemma 2.2 and the non-negativity of function x, we can deduce

$$F_\alpha(t) = x(t) T^{\alpha}_{t_0}(\varphi)(t) \leq 0.$$

Therefore, inequality (3.1) holds. □

Lemma 3.2. Let $\varphi : [t_0, \infty) \to \mathbb{R}$ be monotonically increasing and α-differentiable at a point $t > t_0$. Suppose $x : [t_0, \infty) \to \mathbb{R}$ is a positive and continuously differentiable function at t. Then the inequality

$$T^{\alpha}_{t_0}(\varphi x)(t) \geq \varphi(t) T^{\alpha}_{t_0} x(t), \ \ \forall\, t \geq t_0, \ \ \forall\, \alpha \in (0, 1], \tag{3.2}$$

holds.

Proof. Let the function $\psi(t) = -\varphi(t)$. Then it follows from Lemma 3.1 that

$$T^{\alpha}_{t_0}(\psi x)(t) \leq \psi(t) T^{\alpha}_{t_0} x(t), \ \ \forall\, t \geq t_0, \ \ \forall\, \alpha \in (0, 1].$$

As a consequence, inequality (3.2) holds. □

Lemma 3.3. Let $\varphi : [t_0, \infty) \to \mathbb{R}^n$ be a monotonically decreasing and α-differentiable vector function at a point $t > t_0$. Suppose $x : [t_0, \infty) \to \mathbb{R}^n$ is a positive and continuously differentiable vector function. Then the inequality

$$T^{\alpha}_{t_0}(\varphi^T x)(t) \leq \varphi^T(t) T^{\alpha}_{t_0} x(t), \quad \forall\, t \geq t_0, \quad \forall\, \alpha \in (0, 1],$$

holds.

Proof. Let

$$\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))^T \text{ and } x(t) = (x_1(t), \dots, x_n(t))^T.$$

Since

$$\varphi^T(t)x(t) = \sum_{i=1}^{n} \varphi_i(t) x_i(t),$$

by the linearity of comformable fractional derivative operator, we have

$$T^{\alpha}_{t_0}(\varphi^T x)(t) = \sum_{i=1}^{n} T^{\alpha}_{t_0}(\varphi_i x_i)(t).$$

Note that the functions ϕ_i are monotonically decreasing and α-differentiable at a point $t > t_0$, for $i = 1, 2, \dots, n$, and x_i are positive and continuously differentiable for $i = 1, 2, \dots, n$. Then, applying Lemma 3.1, we get

$$T^{\alpha}_{t_0}(\varphi^T x)(t) \leq \sum_{i=1}^{n} \varphi_i(t) T^{\alpha}_{t_0}(x_i)(t) = \varphi^T(t) T^{\alpha}_{t_0}(x)(t).$$

This completes the proof. □

Lemma 3.4. Let $\phi : [t_0, \infty) \to \mathbb{R}$ be a positive, monotonically decreasing and α-differentiable vector function at a point $t > t_0$. Suppose $x : [t_0, \infty) \to \mathbb{R}$ is a continuously differentiable vector function. Then, for any $t \geq t_0$ and $\alpha \in (0, 1]$, the following inequality holds:

$$T^{\alpha}_{t_0}\big(a\varphi(t)x^p(t) + bx^p(t)\big) \leq p(a\varphi(t) + b)x^{p-1}(t) T^{\alpha}_{t_0} x(t),$$

where a, b are two strictly positive constants and $p = 2^n$, $n \in \mathbb{N}$.

Proof. Since the conformable fractional derivative operator is linear, we have

$$T^{\alpha}_{t_0}\big(a\varphi(t)x^p(t) + bx^p(t)\big) = aT^{\alpha}_{t_0}(\varphi x^p)(t) + bT^{\alpha}_{t_0} x^p(t).$$

Applying Lemmas 3.1 and 2.4, we get

$$\begin{aligned} &T^{\alpha}_{t_0}\big(a\varphi(t)x^p(t) + bx^p(t)\big) \\ &\qquad \leq a\varphi(t) T^{\alpha}_{t_0}(x^p)(t) + bT^{\alpha}_{t_0} x^p(t) \leq (a\varphi(t) + b) T^{\alpha}_{t_0} x^p(t) \leq p(a\varphi(t) + b)x^{p-1}(t) T^{\alpha}_{t_0} x(t). \end{aligned}$$

This completes the proof. □

Assumption 3.1. Let $P : [t_0, \infty) \to \mathbb{R}^{n \times n}$ be an α-differentiable at a point $t > t_0$, symmetric and positive definite matrix function such that

(1) the matrix $P(t) = U(t)D(t)U^T(t)$, where U is α-differentiable at a point $t > t_0$ orthogonal matrix, and $D(t) = \operatorname{diag}(\lambda_{11}(t), \lambda_{22}(t), \dots, \lambda_{nn}(t))$.

(2) the scalar functions $\lambda_{ii}(t)$ are monotonically decreasing and α-differentiable at a point $t > t_0$ for all $i = 1, 2 \dots, n$.

(3) the real valued functions $u_{ij}(t)$ of matrix $U(t)$ are positive, monotonically decreasing and α-differentiable vector functions at a point $t > t_0$ for all $i, j \in \{1, 2, \dots, n\}$.

Lemma 3.5. Let Assumption (3.1) holds. Suppose $x : [t_0, \infty) \to \mathbb{R}^n$ is a continuously differentiable function. Then the inequality

$$T^{\alpha}_{t_0}\big(x^T(t)P(t)x(t)\big) \leq 2x^T(t)P(t)T^{\alpha}_{t_0}(x)(t), \ \ \forall\, \alpha \in (0,1], \ \ \forall\, t \geq t_0,$$

holds.

Proof. By condition (1) of Assumption (3.1), we have $P(t) = U(t)D(t)U(t)^T$. We can write

$$x^T(t)P(t)x(t) = x^T(t)U(t)D(t)U^T(t)x(t).$$

Let $y(t) = U^T(t)x(t)$. Then

$$x^T(t)P(t)x(t) = y^T(t)D(t)y(t) = \sum_{i=1}^{n} \lambda_{ii}(t)y_i^2(t),$$

where $y(t) = (y_1(t), \dots, y_n(t))$. We have

$$T^{\alpha}_{t_0}\big(x^T(t)P(t)x(t)\big) = T^{\alpha}_{t_0}\Big(\sum_{i=1}^{n} \lambda_{ii}(t)y_i^2(t)\Big) = \sum_{i=1}^{n} T^{\alpha}_{t_0}\big(\lambda_{ii}(t)y_i^2(t)\big).$$

Since, by condition (2) of Assumption (3.1), the functions $\lambda_{ii}(t)$ are monotonically decreasing and α-differentiable at a point $t > t_0$ for all $i = 1, 2, \dots, n$, and $y_i^2(t)$ are positive, continuously differentiable functions for $i = 1, 2, \dots, n$, applying Lemma 3.1, we get

$$T^{\alpha}_{t_0}\big(x^T(t)P(t)x(t)\big) \leq \sum_{i=1}^{n} \lambda_{ii}(t)T^{\alpha}_{t_0}y_i^2(t).$$

Now, by using Lemma 2.3, we obtain

$$T^{\alpha}_{t_0}\big(x^T(t)P(t)x(t)\big) \leq 2\sum_{i=1}^{n} \lambda_{ii}(t)y_i(t)T^{\alpha}_{t_0}y_i(t) \leq 2y^T(t)D(t)T^{\alpha}_{t_0}y(t) \leq 2x^T(t)U(t)D(t)T^{\alpha}_{t_0}U^T(t)x(t)$$

Note that under condition (3) of Assumption (3.1), the application of Lemma 3.3 gives

$$T^{\alpha}_{t_0}(U^T(t)x(t)) \leq U^T(t)T^{\alpha}_{t_0}x(t).$$

It follow that

$$T^{\alpha}_{t_0}\big(x^T(t)P(t)x(t)\big) \leq 2x^T(t)U(t)D(t)U^T(t)T^{\alpha}_{t_0}x(t) \leq 2x^T(t)P(t)T^{\alpha}_{t_0}x(t).$$ □

Remark 3.1. If U is constant orthogonal matrix, then we can remove condition (3) for Assumption 3.1 and

$$T^{\alpha}_{t_0}(U^Tx)(t) = U^TT^{\alpha}_{t_0}x(t).$$

3.2 Application to Conformable fractional systems

Example 3.1. Consider the nonautonomous linear conformable fractional order system

$$\begin{cases} T^{\alpha}_0 x_1(t) = -x_1(t) - (1+e^{-t})x_2(t), \\ T^{\alpha}_0 x_2(t) = x_1(t) - x_2(t), \\ T^{\alpha}_0 x_3(t) = -x_3(t), \end{cases}$$

where $0 < \alpha \leq 1$. Let

$$V(t,x) = x_1^2 + (1+e^{-t})x_2^2 + x_3^2, \ \ t \in \mathbb{R}^+, \ \ x = (x_1, x_2, x_3)^T \in \mathbb{R}^3,$$

be the function, which depends on time t and variable x. Note that

$$x_1^2 + x_2^2 + x_3^2 \leq V(t,x) \leq x_1^2 + 2x_2^2 + x_3^2.$$

Then the application of Lemmas 2.2 and 2.3 enables us to calculate the conformable fractional derivative of $V(t,x)$ along the solution $x(t)$ to (2.2) as follows:

$$\begin{aligned} T_0^\alpha V(t,x(t)) &\leq \Big[-2x_1^2(t) - 2(1+e^{-t})x_1(t)x_2(t) \Big] + \Big[2(1+e^{-t})\big(x_1(t)x_2(t) - x_2^2(t)\big) \Big] - 2x_3^2(t) \\ &\leq -2x_1^2(t) - 2x_2^2(t) - 2x_3^2(t) \\ &\leq -2\|x(t)\|^2. \end{aligned}$$

Therefore, we conclude from Theorem 2.2 that the zero solution is globally asymptotically stable (see Figure 3.1).

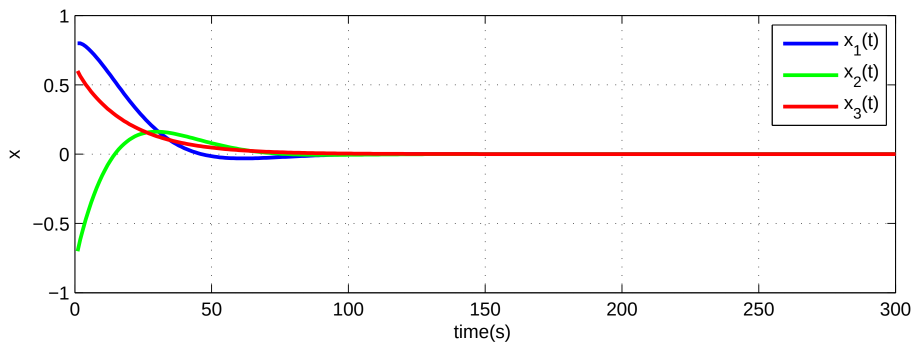

Figure 3.1: Evolution of the states $x_1(t)$, $x_2(t)$ and $x_3(t)$ with the initial conditions $x_1(0) = 0.8$, $x_2(0) = -0.7$, $x_3(0) = 0.6$, and $\alpha = 0.98$.

Example 3.2. Let us consider the nonautonomous nonlinear conformable fractional order system

$$\begin{cases} T_0^\alpha x_1(t) = -x_1(t) + ah(t)x_2^3(t), \\ T_0^\alpha x_2(t) = -x_2(t) - bh(t)x_1^3(t), \end{cases} \tag{3.3}$$

where $0 < \alpha \leq 1$, a, b are the two real strictly positive constants and $h(t)$ is monotonically decreasing, α-differentiable at a point $t > t_0$ and satisfies

$$0 \leq h(t) \leq M.$$

Let us choose the time-varying Lyapunov function

$$V(t,x(t)) = b(1+h(t))x_1^4(t) + a(1+h(t))x_2^4(t), \quad t \in \mathbb{R}^+, \quad x = (x_1,x_2)^T \in \mathbb{R}^2.$$

It is clear that

$$bx_1^4(t) + ax_2^4(t) \leq V(t,x(t)) \leq (1+M)\big(bx_1^4(t) + ax_2^4(t)\big).$$

Then the Conformable fractional derivative of $V(t,x)$ along the solution $x(t)$ to (3.3) is given as follows:

$$\begin{aligned} T_0^\alpha V(t,x(t)) &\leq 4b(1+h(t))x_1^3(t)T_0^\alpha x_1(t) + 4a(1+h(t))x_2^3(t)T_0^\alpha x_2(t) \\ &\leq -4b(1+h(t))x_1^4(t) - 4a(1+h(t))x_2^4(t) \leq -4\big(bx_1^4(t) + ax_2^4(t)\big). \end{aligned}$$

The assumptions of Theorem 2.2 are satisfied with

$$\begin{aligned} \alpha_1(\|x\|) &= bx_1^4(t) + ax_2^4(t), \\ \alpha_2(\|x\|) &= (1+M)\big(bx_1^4(t) + ax_2^4(t)\big), \\ \alpha_3(\|x\|) &= 4\big(bx_1^4(t) + ax_2^4(t)\big). \end{aligned}$$

Hence the zero solution is globally asymptotically stable.

Further, let $B, P \in \mathbb{R}^{n\times n}$ two matrix functions continuous and bounded on $[0,\infty)$.
We denote

$$b = \sup_{t\geq 0} \|B(t)\|, \quad p = \sup_{t\geq 0} \|P(t)\| \text{ and } m = \inf_{t\geq 0} \|P(t)\|.$$

In the sequel, we introduce the following condition.

(H) There exists a constant $\eta > 0$ such that

$$P(t)A(t) + A^T(t)P(t) \leq -\eta I.$$

Here, we are interested in the stability of nonautonomous fractional order system (2.2). So, we consider system (2.2) described in the following form:

$$T^{\alpha}_{t_0} x(t) = A(t)x(t) + F(t, x(t)), \tag{3.4}$$

where $\alpha \in (0,1]$, $x(t) \in \mathbb{R}^n$, $A(t) \in \mathbb{R}^{n\times n}$ is matrix function continuous and bounded on $[0,\infty)$ and $F : [0+\infty[\times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear continuous function which is locally Lipschitz with respect to x and satisfies

$$\|F(t,x)\| \leq \delta\|x\|,$$

where $\delta > 0$ is a positive number.

Theorem 3.1. If there exists a matrix $P(t)$ such that the following conditions are satisfied:

(i) the matrix $P(t)$ satisfies Assumption (3.1);

(ii) **(H)**;

(iii) the number δ satisfying $0 < \delta < \frac{\eta}{2p}$,

then the zero solution of (3.4) is globally asymptotically stable.

Proof. Let us consider the Lyapunov function

$$V(t, x(t)) = x^T(t)P(t)x(t), \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R}^n.$$

The Conformable fractional derivative of $V(t,x)$ along the solution of system (3.4) is

$$\begin{aligned} T^{\alpha}_{t_0} V(t, x(t)) &\leq 2x^T(t)P(t)T^{\alpha}_{t_0} x(t) \\ &\leq x^T(t)\big((P(t)A(t) + A^T(t)P(t)\big)x(t) + 2\big\langle f(t,x), P(t)x(t)\big\rangle \\ &\leq -\eta\|x(t)\|^2 + 2p\delta\|x(t)\|^2 \leq (-\eta + 2p\delta)\|x(t)\|^2 \leq -\gamma_1\|x(t)\|^2, \end{aligned}$$

where $\gamma_1 = \eta - 2p\delta > 0$. Therefore, the conditions of Theorem 2.2 are satisfied. Thus we conclude that the zero solution of system (3.4) is globally asymptotically stable. $\square$

Example 3.3. Consider the fractional order nonlinear system (3.4)

$$T^{\alpha}_0 x(t) = A(t)x(t) + F(t, x(t)),$$

where $x(t) = (x_1(t), x_2(t))^T \in \mathbb{R}^2$,

$$A(t) = \begin{pmatrix} -4 - \dfrac{1}{1+t} & -3 \\ 3 & -5 - \dfrac{1}{1+t} \end{pmatrix} \tag{3.5}$$

and

$$F(t, x(t)) = \Big((1 + \sin(t))\sin(x_2(t)), (1 + \cos(t))\sin(x_1(t))\Big)^T.$$

So, we easily obtain

$$\|F(t,x(t))\| = \sqrt{(1+\sin(t))^2 \sin^2(x_2(t)) + (1+\cos(t))^2 \sin^2(x_1(t))} \le 2\|x(t)\|.$$

Let us consider the continuously differentiable, symmetric, positive definite and bounded matrix

$$P(t) = \begin{pmatrix} \frac{1}{2} + \frac{1}{4}\, e^{-t} & 0 \\ 0 & \frac{1}{2} + \frac{1}{4}\, e^{-t} \end{pmatrix}. \tag{3.6}$$

It is clear that the matrix $P(t)$ satisfies Assumption (3.1). We have

$$P(t)A(t) + A^T(t)P(t) = \begin{pmatrix} h_1(t) & 0 \\ 0 & h_2(t) \end{pmatrix},$$

where

$$h_1(t) = -4 - 2e^{-t} - \frac{1}{1+t} - \frac{1}{2}\,\frac{e^{-t}}{1+t} \ \text{ and } \ h_2(t) = -5 - \frac{5}{2}\, e^{-t} - \frac{1}{1+t} - \frac{1}{2}\,\frac{e^{-t}}{1+t}.$$

We have

$$\sup_{t\ge 0} h_1(t) = -4 \ \text{ and } \ \sup_{t\ge 0} h_2(t) = -5.$$

Therefore, for

$$0 < \mu < \min\Big(-\sup_{t\ge 0} h_1(t), -\sup_{t\ge 0} h_2(t)\Big),$$

the following relationship holds:

$$P(t)A(t) + A^T(t)P(t) \le -\eta I. \tag{3.7}$$

Using the Lyapunov function

$$V(t,x(t)) = x^T(t)P(t)x(t)$$

and if we choose $\mu = \frac{7}{2}$ satisfying (3.7), then we can verify that

$$T_0^\alpha V(t,x(t)) \le -\frac{1}{2}\,\|x(t)\|^2.$$

So, according to Theorem 2.2, the zero solution of system (3.4) is globally asymptotically stable.

Now, we are interested in studying the stabilization of bilinear time-varying control conformable fractional system with a norm-bounded control in the following form:

$$T_{t_0}^\alpha x(t) = A(t)x(t) + u(t)B(t)x(t), \tag{3.8}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times n}$ are the matrix functions continuous and bounded on $[0,\infty)$.

Proposition ([6])**.** Let $B(t)$ and $P(t)$ be bounded matrix functions. Then for $r > 0$, the function

$$g(t,x) = -r\Big(\frac{\|P(t)\|\,\|B(t)\|\,\|x\|}{1+\|P(t)\|\,\|B(t)\|\,\|x\|}\Big)B(t)x$$

is globally Lipschitz with respect to $x \in \mathbb{R}^n$.

Theorem 3.2. Suppose that there exists a matrix $P(t)$ satisfying Assumption 3.1. If we choose $0 < r < \frac{\eta}{2pb}$ and the condition **(H)** is fulfilled, then there exists a bounded feedback law

$$u(t,x) = -r\,\frac{\|P(t)\|\,\|B(t)\|\,\|x\|}{1+\|P(t)\|\,\|B(t)\|\,\|x\|} \tag{3.9}$$

such that the zero solution of closed-loop system (3.8) is fractional exponentially stable.

Proof. Let us consider the Lyapunov function

$$V(t,x(t)) = x^T(t)P(t)x(t), \;\; t\in\mathbb{R}^+, \;\; x\in\mathbb{R}^n.$$

The Conformable fractional derivative of $V(t,x)$ along the solution of the closed-loop system (3.8) by feedback (3.9) is

$$\begin{aligned} T^\alpha_{t_0}V(t,x(t)) &\le 2x^T(t)P(t)T^\alpha_{t_0}x(t) \\ &\le x^T(t)\big((P(t)A(t)+A^T(t)P(t)\big)x(t) + 2u(t)\big\langle B(t)x(t),P(t)x(t)\big\rangle \\ &\le -\eta\|x(t)\|^2 - 2r\,\frac{\|P(t)\|\,\|B\|\,\|x(t)\|}{1+\|P(t)\|\,\|B\|\,\|x(t)\|}\,\big\langle B(t)x(t),P(t)x(t)\big\rangle. \end{aligned}$$

Since

$$\big|\langle B(t)x,P(t)x\rangle\big| \le \|P(t)\|\,\|B(t)\|\,\|x\|^2,$$

we get

$$T^\alpha_{t_0}V(t,x(t)) \le -\eta\|x(t)\|^2 + 2r\|P(t)\|\,\|B(t)\|\,\|x(t)\|^2 \le (-\eta+2rpb)\|x(t)\|^2 \le -\gamma_2\|x(t)\|^2,$$

where $\gamma_2 = \eta - 2rpb > 0$. Also, the matrix P is bounded, then

$$m\|x\|^2 \le V(t,x(t)) \le p\|x\|^2.$$

We can obtain

$$T^\alpha_{t_0}V(t,x(t)) \le -\frac{\gamma_2}{p}\,V(t,x(t)).$$

Using Lemma 2.5, we obtain

$$V(t,x(t)) \le E_\alpha\Big(-\frac{\gamma_2}{p}\,,t-t_0\Big)V(t_0,x(t_0)), \;\; \forall\, t\ge t_0.$$

Therefore,

$$m\|x(t)\|^2 \le V(t,x(t)) \le E_\alpha\Big(-\frac{\gamma_2}{p}\,,t-t_0\Big)V(t_0,x(t_0)) \le pE_\alpha\Big(-\frac{\gamma_2}{p}\,,t-t_0\Big)\|x(t_0)\|^2.$$

Thus it follow that

$$\|x(t)\| \le \sqrt{\frac{p}{m}}\,E_\alpha\Big(-\frac{\gamma_2}{2p}\,,t-t_0\Big)\|x(t_0)\|.$$

Then the zero solution of the closed-loop system (3.8) is fractional exponentially stable. □

Example 3.4. Consider the bilinear time-varying fractional-order system (3.8)

$${}^C D^\alpha_{t_0}x(t) = A(t)x(t) + u(t)B(t)x(t),$$

where $x(t)\in\mathbb{R}^2$, A is of form (3.5) and

$$B(t) = \begin{pmatrix} \frac{1}{4}\,e^{-3t} & 0 \\ 0 & \frac{1}{4}\,e^{-3t} \end{pmatrix}.$$

Let P be given by (3.6) and $\mu = \frac{7}{2}$ satisfies (3.7). Using the Lyapunov function

$$V(t,x(t)) = x^T(t)P(t)x(t)$$

and the feedback function

$$u(t,x) = -\frac{1}{4}\,\frac{(\frac{1}{2}+\frac{1}{4}\,e^{-t})e^{-3t}\|x\|}{4+(\frac{1}{2}+\frac{1}{4}\,e^{-t})e^{-3t}\|x\|}\,,$$

we verify that

$$T^\alpha_0 V(t,x(t)) \le -\frac{1}{2}\,\|x(t)\|^2.$$

Therefore,

$$\|x(t)\| \le \sqrt{\frac{3}{2}}\,E_\alpha\Big(-\frac{1}{3}\,,t\Big)\|x(0)\|.$$

So, the zero solution of closed-loop system (3.8) is fractional exponentially stable (see Figure 3.2).

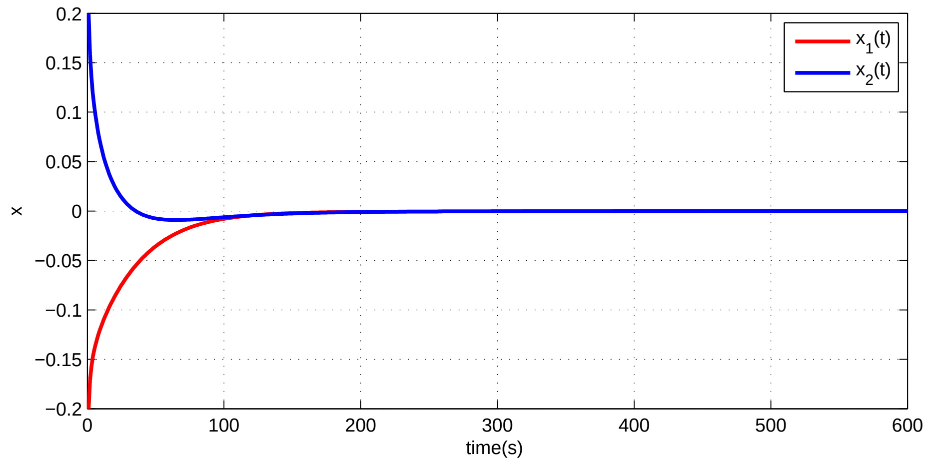

Figure 3.2: Evolution of the states $x_1(t)$ and $x_2(t)$ with feedback (3.9) and initial conditions $x_1(0) = -0.2$, $x_2(0) = 0.2$ and $\alpha = 0.87$.

4 Conclusion

In this paper, we have constructed the continuously differentiable time-varying Lyapunov functions for the stability analysis of nonautonomous conformable fractional-order systems. The stabilization problem and the construction of feedback functions, which make the zero solution of closed-loop bilinear time-varying control conformable fractional systems fractional exponentially stable, have been solved, using the time-varying quadratic Lyapunov function.

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