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ASYMPTOTIC PROPERTIES OF ONE CLASS OF SOLUTIONS OF A BINOMIAL NONAUTONOMOUS FOURTH-ORDER DIFFERENTIAL EQUATION WITH RAPIDLY VARYING NONLINEARITY

Abstract. For a binomial, nonautonomous fourth-order differential equation with rapidly varying nonlinearity, necessary and sufficient conditions for the existence of solutions are obtained in a special case, and asymptotic representations of these solutions and their derivatives up to the third order inclusive are established as $t \uparrow \omega$ $(\omega \le +\infty)$.

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1 Introduction

We consider the differential equation

$$y^{(4)} = \alpha_0 p(t)\varphi(y), \tag{1.1}$$

where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\to]0,+\infty[$ is acontinuous function, $-\infty < a < \omega \le +\infty$, $\varphi : \Delta_{Y_0} \to]0,+\infty[$ is a twice continuously differentiable function such that

$$\varphi'(y) \neq 0 \ \text{ at } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi(y) = \begin{cases} \text{either} & 0, \\ \text{or} & +\infty, \end{cases} \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi(y)\varphi''(y)}{\varphi'^2(y)} = 1, \tag{1.2}$$

Y_0 equals either 0, or $\pm\infty$, Δ_{Y_0} is a one-sided neighborhood of Y_0.

From conditions (1.2) directly follows that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)} \ \text{ as } \ y \to Y_0 \ \ (y \in \Delta_{Y_0}) \ \text{ and } \ \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty. \tag{1.3}$$

According to these conditions, the function φ and its first-order derivative (see the monograph by M. Maric [8, Chapter 3, § 3.4, Lemmas 3.2, 3.3, pp. 91–92]) are rapidly varying functions as $y \to Y_0$.

Definition 1.1. A solution y of the differential equation (1.1) is called a $P_\omega(Y_0,\lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on the interval $[t_0,\omega[\subset[a,\omega[$ and satisfies the following conditions:

$$y(t) \in \Delta_{Y_0} \ \text{ at } \ t \in [t_0,\omega[\,, \quad \lim_{t\uparrow\omega} y(t) = Y_0,$$

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{or} & 0, \\ \text{or} & \pm\infty, \end{cases} \quad (k=1,2,3), \quad \lim_{t\uparrow\omega} \frac{[y'''(t)]^2}{y''(t)y^{(4)}(t)} = \lambda_0.$$

Earlier, the asymptotic behaviour of $P_\omega(Y_0,\lambda_0)$-solutions of equation (1.1) was investigated in [5] in the case where $\lambda_0 \in \mathbb{R}\setminus\{0,\frac{1}{2},\frac{2}{3},1\}$. The aim of this paper is to study the existence and asymptotic behaviour of $P_\omega(Y_0,\lambda_0)$-solutions in the special case where $\lambda_0 = 1$. In this case, due to the a priori asymptotic properties of $P_\omega(Y_0,\lambda_0)$-solutions (see [2, Chapter 3, § 10]), for each $P_\omega(Y_0,1)$-solution, the following asymptotic properties hold:

$$\frac{y'(t)}{y(t)} \sim \frac{y''(t)}{y'(t)} \sim \frac{y'''(t)}{y''(t)} \sim \frac{y''''(t)}{y'''(t)} \ \text{ as } \ t\uparrow\omega, \ \text{ and } \ \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = \pm\infty, \tag{1.4}$$

where

$$\pi_\omega(t) = \begin{cases} t & \text{if } \ \omega = +\infty, \\ t-\omega & \text{if } \ \omega < +\infty. \end{cases}$$

From this, in particular, tt follows that the $P_\omega(Y_0,1)$-solution of equation (1.1) and its derivatives up to the third order inclusive are rapidly varying functions as $t\uparrow\omega$.

2 Some auxiliary statements

In studying the behaviour of $P_\omega(Y_0,1)$-solutions, along with (1.4), we will use some properties of functions of the class $\Gamma_{Y_0}(Z_0)$. This class of functions was introduced in the work of A. G. Chernikova [1] using the well-known class Γ introduced by L. de Haan. A detailed description of the class Γ can be found in the monograph by N. H. Bingham, C. M. Goldie, J. L. Teugels [4, Section 3.10, pp. 174–180].

Definition 2.1. The class $\Gamma_{Y_0}(Z_0)$ consists of the set of continuous and monotonic functions $f : \Delta_{Y_0} \to]0,+\infty[$ satisfying the condition

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}(b)}} f(y) = Z_0 \in \{0;+\infty\} \tag{2.1}$$

for each of which the corresponding function f_0 defined below is continuous and non-decreasing:

(1) the function $f_0(y) = \frac{1}{f(y)}$ for $Y_0 = +\infty$ and $Z_0 = 0$;

(2) the function $f_0(y) = f(-y)$ for $Y_0 = -\infty$ and $Z_0 = +\infty$;

(3) the function $f_0(y) = f\left(\frac{1}{y}\right)$ for $Y_0 = 0$, $\Delta_{Y_0} =]0, y_0]$ and $Z_0 = +\infty$;

(4) the function $f_0(y) = \frac{1}{f(\frac{1}{y})}$ for $Y_0 = 0$, $\Delta_{Y_0} =]0, y_0]$ and $Z_0 = 0$;

(5) the function $f_0(y) = f(-\frac{1}{y})$ for $Y_0 = 0$, $\Delta_{Y_0} = [y_0, 0[$ and $Z_0 = +\infty$;

(6) the function $f_0(y) = \frac{1}{f(-\frac{1}{y})}$ for $Y_0 = 0$, $\Delta_{Y_0} = [y_0, 0[$ and $Z_0 = 0$;

(7) the function $f_0(y) \equiv f(y)$ for $Y_0 = +\infty$ and $Z_0 = +\infty$,

and each such function f_0 belongs to the class Γ.

From this definition and the properties of a function of the class Γ, it is clear that for every function f of the class $\Gamma_{Y_0}(Z_0)$, there exists a continuous and nonzero function $g : \Delta_{Y_0} \to \mathbb{R} \setminus \{0\}$, which is called auxiliary to f, such that

$$\lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{f(y + ug(y))}{f(y)} = e^u \quad \text{for any} \quad u \in \mathbb{R}. \tag{2.2}$$

This auxiliary function g is uniquely defined up to the equivalent functions for $y \to Y_0$, one of which is the function

$$\frac{\int\limits_Y^y f(x)\,dx}{f(y)}, \quad \text{where} \quad Y = \begin{cases} y_0, & \text{if} \quad \lim\limits_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} f(y) = +\infty, \\ Y_0, & \text{if} \quad \lim\limits_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} f(y) = 0. \end{cases} \tag{2.3}$$

In addition, from the properties of the functions of the class $\Gamma_{Y_0}(Z_0)$ established in the work of A. G. Chernikova [1], the following auxiliary statements hold.

Lemma 2.1.

(1) *If $f \in \Gamma_{Y_0}(Z_0)$, then it is fast changing as $y \to Y_0$, and its auxiliary function g satisfies the condition*

$$\lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{g(y)}{y} = 0.$$

(2) *If $f \in \Gamma_{Y_0}(Z_0)$ with an auxiliary function g, then, for any continuous function $u : \Delta_{Y_0} \to \mathbb{R}$ satisfying the conditions*

$$\lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} u(y) = u_0 \in \mathbb{R}, \qquad \lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} f(y + u(y)g(y)) = Z_0,$$

there is a limiting ratio

$$\lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{f(y + u(y)g(y))}{f(y)} = e^{u_0}.$$

Lemma 2.2. *If $f \in \Gamma_{Y_0}(Z_0)$ with an auxiliary function g is strictly monotonic, then its inverse $f^{-1} : \Delta_{Z_0} \to \Delta_{Y_0}$ is a slowly varying function as $z \to Z_0$ and satisfies the asymptotic relation*

$$\lim_{\substack{z\to Z_0 \\ z\in\Delta_{Z_0}}} \frac{f^{-1}(\lambda z) - f^{-1}(z)}{g(f^{-1}(z))} = \ln\lambda \quad \textit{for any} \quad \lambda > 0, \tag{2.4}$$

and moreover, for each $\Lambda > 1$, this asymptotic relation is satisfied uniformly by $\lambda \in [\frac{1}{\Lambda}, \Lambda]$.

Lemma 2.3. *If the function $f : \Delta_{Y_0} \to]0, +\infty[$ is twice continuously differentiable and satisfies the conditions*

$$f'(y) \neq 0, \qquad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} f(y) = Z_0 \in \{0; +\infty\}, \qquad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{f''(y) f(y)}{f'^2(y)} = 1,$$

then the functions f and f' belong to the class $\Gamma_{Y_0}(Z_0)$ with an auxiliary function g, which is uniquely determined up to equivalence as $y \to Y_0$. Moreover, one of the following functions can be chosen as g:

$$\frac{\int\limits_{Y}^{y} f(x)\, dx}{f(y)} \sim \frac{f(y)}{f'(y)} \sim \frac{f'(y)}{f''(y)},$$

where Y is defined in (2.3).

In addition to these lemmas, we also provide a statement that belongs to the theory of slowly, properly and rapidly varying functions (see, e.g., the monograph [4, Appendix, Proposition 10, p. 117]).

Lemma 2.4. *If $f_0 : \Delta_{Y_0} \to \mathbb{R} \setminus \{0\}$ is a continuously differentiable function and*

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{y f_0'(y)}{f_0(y)} = \sigma, \tag{2.5}$$

then the function f_0 is, respectively, slowly varying function in the case $\sigma = 0$, a regularly varying function in the case of $0 < |\sigma| < +\infty$, and a rapidly varying function in the case of $\sigma = \pm\infty$, as $y \to Y_0$.

For more information on slowly, regularly and rapidly varying functions, as well as their properties, see monographs [2,4,8,10]. In particular, it is known that for every regularly varying as $y \to Y_0$ function $f : \Delta_{Y_0} \to \mathbb{R} \setminus \{0\}$ of order σ, the following representation holds:

$$f(y) = |y|^{\sigma} L(y), \tag{2.6}$$

where $L : \Delta_{Y_0} \to \mathbb{R} \setminus \{0\}$ is a slowly varying function as $y \to Y_0$, i.e., one for which the asymptotic relation

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{L(y\lambda)}{L(y)} = 1 \text{ for any } \lambda > 0 \tag{2.7}$$

is satisfied, and which is uniform with respect to λ for any finite interval $[c, d] \in]0, +\infty[$. In addition, there exists an equivalent continuously differentiable function f_0 (called a normalised regularly varying function of order σ) as $y \to Y_0$ for which condition (2.5) is satisfied. In addition to the above statements, we also use Theorem 2.2 from [6] which concerns the existence of vanishing solutions at a special point for a system of quasilinear differential equations.

3 Main results

Let us introduce the necessary auxiliary notation and assume that the domain of the function φ in equation (1.1) is defined as follows:

$$\Delta_{Y_0} = \Delta_{Y_0}(y_0), \text{ where } \Delta_{Y_0}(y_0) = \begin{cases} [y_0, Y_0[& \text{if } \Delta_{Y_0} \text{ left side } Y_0, \\ ]Y_0, y_0] & \text{if } \Delta_{Y_0} \text{ right side } Y_0, \end{cases}$$

where $y_0 \in \Delta_{Y_0}$ such that $|y_0| < 1$ for $Y_0 = 0$ and $y_0 > 1$ ($y_0 < -1$) for $Y_0 = +\infty$ (for $Y_0 = -\infty$).

Let us put

$$\mu_0 = \operatorname{sign} \varphi'(y), \quad \nu_0 = \operatorname{sign} y_0, \quad \nu_1 = \begin{cases} 1 & \text{if } \Delta_{Y_0} = [y_0, Y_0[\,, \\ -1 & \text{if } \Delta_{Y_0} =]Y_0, y_0]. \end{cases} \tag{3.1}$$

Note that the numbers ν_0, ν_1 determine the signs of any $P_\omega(Y_0,\lambda_0)$-solution, the first derivative in some left neighbourhood ω. In this case, the conditions

$$\nu_0\nu_1<0 \ \text{ if } \ Y_0=0, \quad \nu_0\nu_1>0 \ \text{ if } \ Y_0=\pm\infty,$$

are necessary for the existence of such solutions.

In addition, let us introduce two auxiliary functions

$$J_0(t)=\int\limits_{A_0}^{t} p_0^{\frac14}(\tau)\,d\tau \ \text{ and } \ \Phi(y)=\int\limits_{B}^{y}\frac{ds}{|s|^{\frac34}\varphi^{\frac14}(s)}\,,$$

where $p_0:[a,\omega[\,\to\,]0,+\infty[$ is a continuous or continuously differentiable function such that $p(t)\sim p_0(t)$ as $t\uparrow\omega$,

$$A_0=\begin{cases}\omega, & \text{if } \displaystyle\int\limits_a^{\omega} p_0^{\frac14}(\tau)\,d\tau<+\infty,\\ a, & \text{if } \displaystyle\int\limits_a^{\omega} p_0^{\frac14}(\tau)\,d\tau=+\infty,\end{cases}\qquad B=\begin{cases}Y_0, & \text{if } \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{|s|^{\frac34}\varphi^{\frac14}(s)}=const,\\ y_0, & \text{if } \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{|s|^{\frac34}\varphi^{\frac14}(s)}=\pm\infty.\end{cases}$$

Now, let us consider some properties of the function Φ. It preserves the sign on Δ_{Y_0}, tends either to zero or $\pm\infty$ as $y\to Y_0$, and is increasing on Δ_{Y_0}, since on this interval, $\Phi'(t)=|y|^{-\frac34}\varphi^{-\frac14}(y)>0$. Therefore, there exists an inverse function $\Phi^{-1}:\Delta_{Z_0}\to\Delta_{Y_0}$, where, due to the second condition (1.2) and the monotonic growth of Φ^{-1},

$$\begin{gathered}Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi(y)=\begin{cases}\text{either} & 0,\\ \text{or} & \pm\infty,\end{cases}\\ \Delta_{Z_0}=\begin{cases}[z_0,Z_0[\,, & \text{if } \ \Delta_{Y_0}=[y_0,Y_0[\,,\\ ]Z_0,z_0], & \text{if } \ \Delta_{Y_0}=]Y_0,y_0],\end{cases}\qquad z_0=\Phi(y_0).\end{gathered}\tag{3.2}$$

In addition, we would like to draw your attention to the fact that

$$\Big(\frac{\varphi^{\frac34}(y)}{|y|^{\frac34}\varphi'(y)}\Big)'=\frac{1}{|y|^{\frac34}\varphi^{\frac14}(y)}\Big[\frac34-\frac34\,\frac{\varphi(y)}{y\varphi'(y)}-\frac{\varphi(y)\varphi''(y)}{\varphi'^2(y)}\Big].$$

Hence, taking into account conditions (1.2) and (1.3), we obtain the following relation:

$$\Big(\frac{\varphi^{\frac34}(y)}{|y|^{\frac34}\varphi'(y)}\Big)'=\frac{1}{|y|^{\frac34}\varphi^{\frac14}(y)}\Big[-\frac14+o(1)\Big] \ \text{ as } \ y\to Y_0.$$

Integrating this ratio over the interval from y_0 to y and taking into account the rule for choosing the integration limit B in the function Φ, we conclude that

$$\Phi(y)=-\frac{4\varphi^{\frac34}(y)}{|y|^{\frac34}\varphi'(y)}\,[1+o(1)] \ \text{ as } \ y\to Y_0.\tag{3.3}$$

Hence, taking into account the sign φ', we obtain

$$\operatorname{sign}\Phi(y)=-\mu_0 \ \text{ at } \ y\in\Delta_{Y_0}.\tag{3.4}$$

Considering (3.3) and (1.3), we have

$$\begin{gathered}\frac{\Phi'(y)}{\Phi(y)}=\frac{|y|^{-\frac34}\varphi^{-\frac14}(y)}{\Phi(y)}\sim-\frac{\varphi'(y)}{4\varphi(y)} \ \text{ as } \ y\to Y_0,\\ \frac{\Phi''(y)\Phi(y)}{\Phi'^2(y)}=-\frac14\,\frac{|y|^{-\frac14}\varphi^{-\frac74}(y)\Big[\frac{\phi(y)}{y\varphi'(y)}+1\Big]\Phi(y)}{|y|^{-1}\varphi^{-1}(y)}\sim 1 \ \text{ as } \ y\to Y_0.\end{gathered}$$

Therefore, $\Phi \in \Gamma_{Y_0}(Z_0)$ and, according to (2.2) and (2.3), one of the equivalent functions can be chosen as its complement

$$\frac{\Phi'(y)}{\Phi''(y)} \sim \frac{\Phi(y)}{\Phi'(y)} \sim -\frac{4\varphi(y)}{\varphi'(y)} \quad \text{as} \quad y \to Y_0.$$

Since

$$\lim_{z\to Z_0} \frac{z(\varphi(\Phi^{-1}(z)))'}{\varphi(\Phi(z))} = \lim_{z\to Z_0} \frac{z(\varphi'(\Phi^{-1}(z)))|\Phi(z)|^{\frac{3}{4}}\varphi^{\frac{1}{4}}(\Phi^{-1}(z))}{\varphi(\Phi^{-1}(z))} = \lim_{y\to Y_0} \frac{\Phi(y)\varphi'(y)|y|^{\frac{3}{4}}}{\varphi^{\frac{3}{4}}(y)} = -4,$$

according to Lemma 2.3, the function $\varphi(\Phi^{-1}(z))$ is a regularly varying function of order -4 as $z \to Z_0$, i.e., there exists a representation

$$\varphi(\Phi^{-1}(z)) = |z|^{-4}L(z),$$

where $L : \Delta_{Z_0} \to]0, +\infty[$ is a slowly varying function as $z \to Z_0$. Similarly, it can be shown that the function $\varphi'(\Phi^{-1}(z))$ is also regularly varying of order -4 as $z \to Z_0$. In addition to the above notation, we also introduce the following auxiliary functions:

$$q(t) = \frac{(\Phi^{-1}(\nu_1 J_0(t)))'}{\alpha_0 J_3(t)}, \quad H(t) = \frac{\Phi^{-1}(\nu_1 J_0(t))\varphi'(\Phi^{-1}(\nu_1 J_0(t)))}{\varphi(\Phi^{-1}(\nu_1 J_0(t)))},$$

$$J_1(t) = \int_{A_1}^{t} p_0(\tau)\varphi(\Phi^{-1}(\nu_1 J_0(\tau)))\,d\tau, \quad J_2(t) = \int_{A_2}^{t} J_1(\tau)\,d\tau, \quad J_3(t) = \int_{A_3}^{t} J_2(\tau)\,d\tau,$$

where

$$A_1 = \begin{cases} t_1 & \text{if } \displaystyle\int_{t_1}^{\omega} p_0(\tau)\varphi(\Phi^{-1}(\nu_1 J_0(\tau)))\,d\tau = +\infty, \\ \omega & \text{if } \displaystyle\int_{t_1}^{\omega} p_0(\tau)\varphi(\Phi^{-1}(\nu_1 J_0(\tau)))\,d\tau < +\infty, \quad t_1 \in [a, \omega], \end{cases}$$

$$A_2 = \begin{cases} t_1 & \text{if } \displaystyle\int_{t_1}^{\omega} J_1(\tau)\,d\tau = +\infty, \\ \omega & \text{if } \displaystyle\int_{t_1}^{\omega} J_1(\tau)\,d\tau < +\infty, \end{cases} \qquad A_3 = \begin{cases} t_1 & \text{if } \displaystyle\int_{t_1}^{\omega} J_2(\tau)\,d\tau = +\infty, \\ \omega & \text{if } \displaystyle\int_{t_1}^{\omega} J_2(\tau)\,d\tau < +\infty. \end{cases}$$

The following statement is true for equation (1.1).

Theorem 3.1. *For the existence of $P_\omega(Y_0, 1)$-solutions of differential equation (1.1), the following inequalities*

$$\nu_1\mu_0 J_0(t) < 0 \ \text{ for } \ t \in]a, \omega[, \tag{3.5}$$

$$\alpha_0\nu_1 < 0 \ \text{ if } \ Y_0 = 0, \quad \alpha_0\nu_1 > 0 \ \text{ if } \ Y_0 = \pm\infty, \tag{3.6}$$

and conditions

$$\frac{\alpha_0 J_3(t)}{\Phi^{-1}(\nu_1 J_0(t))} \sim \frac{J_1'(t)}{J_1(t)} \sim \frac{J_2'(t)}{J_2(t)} \sim \frac{J_3'(t)}{J_3(t)} \sim \frac{(\Phi^{-1}(\nu_1 J_0(t)))'}{\Phi^{-1}(\nu_1 J_0(t))} \quad \text{as } t \uparrow \omega, \tag{3.7}$$

$$\lim_{t\uparrow\omega} H(t) = \pm\infty, \quad \nu_1 \lim_{t\uparrow\omega} J_0(t) = Z_0,$$

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)(\Phi^{-1}(\nu_1 J_0(t)))'}{\Phi^{-1}(\nu_1 J_0(t))} = \pm\infty, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_0'(t)}{J_0(t)} = \pm\infty \tag{3.8}$$

must be satisfied.cIn addition, for each such solution, as $t \uparrow \omega$, the following asymptotic representations hold:

$$y(t) = \Phi^{-1}(\nu_1 J_0(t))\Big[1 + \frac{o(1)}{H(t)}\Big], \tag{3.9}$$

$$y'(t) = \alpha_0 J_3(t)[1+o(1)], \quad y''(t) = \alpha_0 J_2(t)[1+o(1)], \quad y'''(t) = \alpha_0 J_1(t)[1+o(1)] \tag{3.10}$$

Proof. Let $y : [t_0, \omega[\to \mathbb{R}$ be arbitrary and $P_\omega(Y_0, 1)$ be a solution of differential equation (1.1). Then, according to (1.1), conditions (1.4) and the introduced notation, we have

$$\operatorname{sign} y(t) = \alpha_0, \quad \operatorname{sign} y'(t) = \nu_1, \quad \operatorname{sign} y''(t) = \alpha_0,$$
$$\operatorname{sign} y'''(t) = \nu_1, \quad \operatorname{sign} y^{(4)}(t) = \alpha_0.$$

At the same time, as already established, condition (3.6) is satisfied. Furthermore, according to the a priori properties of $P_\omega(Y_0, \lambda_0)$-solutions (1.4), the following asymptotic relation holds

$$y''''(t) = \frac{y''''(t)}{y'''(t)}\,\frac{y'''(t)}{y''(t)}\,\frac{y''(t)}{y'(t)}\,\frac{y'(t)}{y(t)}y(t) \sim \Big(\frac{y'(t)}{y(t)}\Big)^4 y(t) \ \text{ as } \ t \uparrow \omega.$$

From this ratio, (1.1) and taking into account that

$$\operatorname{sign}\Big(\frac{y'(t)}{y(t)}\Big) = \nu_1\alpha_0,$$

we obtain

$$\Big(\frac{y'(t)}{y(t)}\Big)^4 = \alpha_0 p_0(t)\,\frac{\varphi(y(t))}{y(t)}\,[1+o(1)] \ \text{ as } \ t \uparrow \omega$$

and therefore,

$$\frac{y'(t)}{y(t)} = \nu_1\alpha_0\Big(p_0(t)\,\frac{\varphi(y(t))}{|y(t)|}\Big)^{\frac14}[1+o(1)] \ \text{ as } \ t \uparrow \omega, \tag{3.11}$$
$$\frac{y'(t)}{|y(t)|} = \nu_1\Big(p_0(t)\,\frac{\varphi(y(t))}{|y(t)|}\Big)^{\frac14}[1+o(1)] \ \text{ as } \ t \uparrow \omega,$$

or

$$\frac{y'(t)}{|y(t)|^{\frac34}\varphi^{\frac14}(t)} = \nu_1 p_0^{\frac14}(t)[1+o(1)] \ \text{ as } \ t \uparrow \omega.$$

Integrating this ratio over the interval from t_1 to t, we obtain

$$\int\limits_{y(t_1)}^{y(t)} \frac{ds}{|s|^{\frac34}\varphi^{\frac14}(s)} = \nu_1 \int\limits_{t_1}^{t} p_0^{\frac14}(\tau)[1+o(1)]\,d\tau \ \text{ as } \ t \uparrow \omega.$$

Since, according to the definition of $P_\omega(Y_0, 1)$-the solutions, $y(t) \to Y_0$ as $t \uparrow \omega$, it follows that the improper integrals $\int\limits_{y(t_1)}^{Y_0} \frac{ds}{|s|^{\frac34}\varphi^{\frac14}(s)}$ and $\int\limits_{t_1}^{t} p_0^{\frac14}(\tau)\,d\tau$ coincide or diverge simultaneously. Given this fact and the rule for choosing the limits of integration A_0 and B in the functions J_0 and Φ introduced at the beginning of this section, the above relation can be written as

$$\Phi(y(t)) = \nu_1 J_0(t)[1+o(1)] \ \text{ as } \ t \uparrow \omega. \tag{3.12}$$

Hence, taking into account conditions (3.1) and (3.4), it follows that inequality (3.5) and the second of conditions (3.8) are satisfied. Moreover, using (3.11) and (3.12), we obtain the relation

$$\frac{y'(t)}{|y(t)|^{\frac34}\varphi^{\frac14}(t)\Phi(y(t))} = \frac{p_0^{\frac14}(t)}{J_0(t)}\,[1+o(1)] \ \text{ as } \ t \uparrow \omega.$$

Applying relation (3.3) for this purpose, we obtain

$$-\frac{y'(t)\varphi'(y(t))}{4\varphi(t)} = \frac{p_0^{\frac{1}{4}}(t)}{J_0(t)}\,[1+o(1)] \ \text{ as } \ t\uparrow\omega.$$

Next, we have

$$\frac{\pi_\omega(t)y'(t)}{y(t)}\,\frac{y(t)\varphi'(y(t))}{\varphi(y(t))} = -\frac{4\pi_\omega(t)J_0'(t)}{J_0(t)}\,[1+o(1)] \ \text{ as } \ t\uparrow\omega,$$

whence, taking into account the second of the asymptotic relations (1.3), (1.4), the validity of the fourth of conditions (3.8) follows.

Using the fact that there is an inverse function to the function Φ and (3.12), we obtain

$$y(t) = \Phi^{-1}(\nu_1 J_0(t)[1+o(1)]). \tag{3.13}$$

Given that the second condition (3.8) is satisfied, we find that the function $\Phi^{-1}(z)$ is slowly varying, and $\varphi(\Phi^{-1}(z))$ is a regularly varying function of order -4 as $z\to Z_0$, then according to the uniform convergence theorem for slowly varying functions, we have

$$\Phi^{-1}\big(\nu_1 J_0(t)[1+o(1)]\big) \sim \Phi^{-1}(\nu_1 J_0(t)) \ \text{ as } \ t\uparrow\omega,$$
$$\varphi\big(\Phi^{-1}(\nu_1 J_0(t)[1+o(1)])\big) \sim \varphi\big(\Phi^{-1}(\nu_1 J_0(t))\big) \ \text{ as } \ t\uparrow\omega.$$

Based on these asymptotic relations and taking into account (1.1), (3.11) and (3.13), it follows that

$$\frac{y'(t)}{y(t)} \sim \alpha_0\nu_1\Big(\frac{p_0(t)\varphi(\Phi^{-1}(\nu_1 J_0(t)))}{\Phi^{-1}(\nu_1 J_0(t))}\Big)^{\frac{1}{4}} \ \text{ as } \ t\uparrow\omega.$$

Since $y(t)\sim\Phi^{-1}(\nu_1 J_0(t))$, then $\ln y(t)\sim\ln\Phi^{-1}(\nu_1 J_0(t))$ as $t\uparrow\omega$. Differentiating both parts, we obtain

$$\frac{y'(t)}{y(t)} \sim \frac{(\Phi^{-1}(\nu_1 J_0(t)))'}{\Phi^{-1}(\nu_1 J_0(t))} \ \text{ as } \ t\uparrow\omega.$$

Comparing the latter expression with the previous one, we get

$$\frac{y'(t)}{y(t)} \sim \alpha_0\nu_1\Big(\frac{p_0(t)\varphi(\Phi^{-1}(\nu_1 J_0(t)))}{\Phi^{-1}(\nu_1 J_0(t))}\Big)^{\frac{1}{4}} = \frac{(\Phi^{-1}(\nu_1 J_0(t)))'}{\Phi^{-1}(\nu_1 J_0(t))} \ \text{ as } \ t\uparrow\omega. \tag{3.14}$$

Substituting (3.13) into the right-hand side of equation (1.1), we obtain

$$y^{(4)}(t) = \alpha_0 p_0(t)\varphi(\Phi^{-1}(\nu_1 J_0(t)))[1+o(1)] \ \text{ as } \ t\uparrow\omega.$$

Next, integrating successively the last relation over the interval from t_2 to t, where $t_2\in[t_1,\omega[$ is chosen so that $\nu_1 J_0(t)\in\Delta Z_0$ for $t\in[t_2,\omega[$, and taking into account the definition of the $P_\omega(Y_0,1)$-solution, we obtain the following asymptotic representations:

$$y'''(t) = \alpha_0 J_1(t))[1+o(1)] \ \text{ as } \ t\uparrow\omega,$$
$$y''(t) = \alpha_0 J_2(t))[1+o(1)] \ \text{ as } \ t\uparrow\omega,$$
$$y'(t) = \alpha_0 J_3(t))[1+o(1)] \ \text{ as } \ t\uparrow\omega,$$

i.e., reflections (3.10) occur. Taking into account (3.14) and based on (1.4), we obtain that asymptotic relations (3.7) hold. Morover, based on condition (1.4), the third condition (3.8) is satisfied. The validity of the asymptotic representation (3.9) follows directly from (3.13) and Lemma 2.2, if we take into account that $\Phi\in\Gamma(Y_0,Z_0)$ with the auxiliary function $g(y) = -\frac{4\varphi(y)}{\varphi'(y)}$. □

To establish the actual existence of solutions with the obtained asymptotic representations, in addition to the necessary conditions from Theorem 3.1, some auxiliary statements will be needed.

Lemma 3.1. *Assume that for the function*

$$l(y)=\frac{(\frac{\varphi'(y)}{\varphi(y)})'}{(\frac{\varphi'(y)}{\varphi(y)})^2}\left|\frac{y\varphi'(y)}{\varphi(y)}\right|^{\frac{3}{4}}$$

there is a finite or equal to $\pm\infty$ limit. Then this limit can only be zero. That is,

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{(\frac{\varphi'(y)}{\varphi(y)})'}{(\frac{\varphi'(y)}{\varphi(y)})^2}\left|\frac{y\varphi'(y)}{\varphi(y)}\right|^{\frac{3}{4}}=0. \tag{3.15}$$

Proof. Let us assume the opposite. Then the following relation holds:

$$\frac{(\frac{\varphi'(y)}{\varphi(y)})'}{|\frac{\varphi'(y)}{\varphi(y)}|^{\frac{5}{4}}}=\frac{l(y)}{|y|^{\frac{3}{4}}},$$

in which

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} l(y)=\begin{cases}\text{either } \ const\neq 0,\\ \text{or } \ \pm\infty\end{cases} \tag{3.16}$$

Integrating this ratio over the interval from y_0 to y, where $y_0, y\in\Delta_{Y_0}$, we obtain

$$\mu_0\left(\left|\frac{\varphi'(y)}{\varphi(y)}\right|^{-\frac{1}{4}}-\left|\frac{\varphi'(y_0)}{\varphi(y_0)}\right|^{-\frac{1}{4}}\right)=\int_{y_0}^{y}\frac{l(s)}{|s|^{\frac{3}{4}}}\,ds. \tag{3.17}$$

Next, consider two possible cases.

1) Let us first assume that $\int\limits_{y_0}^{Y_0}\frac{l(s)}{|s|^{\frac{3}{4}}}\,ds=\pm\infty$. In this case, (3.17) can be written as follows:

$$\mu_0\left|\frac{y\varphi'(y)}{\varphi(y)}\right|^{-\frac{1}{4}}=\frac{\int\limits_{y_0}^{y}\frac{l(s)}{|s|^{\frac{3}{4}}}\,ds}{|y|^{\frac{1}{4}}}[1+o(1)] \ \text{ as } \ y\to Y_0. \tag{3.18}$$

Here, according to L'Hôpital's rule in Stolz form,

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\int\limits_{y_0}^{y}\frac{l(s)}{|s|^{\frac{3}{4}}}\,ds}{|y|^{\frac{1}{4}}}=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} 4\nu_0 l(y)$$

and therefore, based on (3.16), the right-hand side of (3.18) tends either to a constant value other than zero or to infinity as $y\to Y_0$. At the same time, according to the second condition (1.3), the left-hand side of (3.18) tends to zero as $y\to Y_0$, i.e., we obtain a contradiction.

2) Let now $\int\limits_{y_0}^{Y_0}\frac{l(s)}{|s|^{\frac{3}{4}}}\,ds=const$, which is possible only for $Y_0=0$. In this case, from (3.17) we have

$$\mu_0\left|\frac{\varphi'(y)}{\varphi(y)}\right|^{-\frac{1}{4}}=c+\int_0^{y}\frac{l(s)}{|s|^{\frac{3}{4}}}\,ds,$$

where c is a real constant. We will show that here $c=0$. If this is not the case, then we will have the relation

$$\frac{\varphi'(y)}{\varphi(y)}=c_1+o(1) \ \text{ as } \ y\to 0, \ \text{ where } \ c_1\neq 0,$$

from which it follows that

$$\lim_{\substack{y\to 0\\ y\in\Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = 0.$$

However, this contradicts the second condition (1.3).

Thus, in case 2), we have

$$\mu_0\Big|\frac{\varphi'(y)}{\varphi(y)}\Big|^{-\frac{1}{4}} = \int\limits_0^y \frac{l(s)}{|s|^{\frac{3}{4}}}\,ds,$$

from which we find that

$$\mu_0\Big|\frac{y\varphi'(y)}{\varphi(y)}\Big|^{-\frac{1}{4}} = \frac{\int\limits_0^y \frac{l(s)}{|s|^{\frac{3}{4}}}\,ds}{|y|^{\frac{1}{4}}}.$$

Here, the limit of the right-hand side, as $y\to 0$, based on L'Hôpital's rule and (3.16) is either a constant different from zero or equal to $\pm\infty$, whereas the left-hand side, according to the second condition (1.3), is equal to zero. The results obtained in each of the two possible cases indicate that the assumption of a non-zero limit of the function $l(y)$ as $y\to Y_0$ was incorrect, and therefore, the boundary condition (3.15) holds. □

Lemma 3.2. *Assume for the function*

$$\psi(t) = \frac{q'(t)|H|^{\frac{1}{2}}(t)J_3(t)}{J_3'(t)}$$

there is a finite or equal to $\pm\infty$ limit. Then this limit can only be 0. That is,

$$\lim_{t\uparrow\omega} \frac{q'(t)|H|^{\frac{1}{2}}(t)J_3(t)}{J_3'(t)} = 0. \tag{3.19}$$

Proof. If a finite or equal $\pm\infty$ limit exists on the left-hand side of the limit relation (3.19), this limit is zero if the following condition is satisfied:

$$\int\limits_{y_0}^{Y_0} \Big(\frac{y\varphi'(y)}{\varphi(y)}\Big)^{-\frac{1}{2}}\,\frac{dy}{y} = \pm\infty. \tag{3.20}$$

Using relation (3.14), we have

$$\frac{(\Phi^{-1}(\nu_1 J_0(t)))'}{\Phi^{-1}(\nu_1 J_0(t))} \sim \nu_1\alpha_0\Big(\frac{p_0(t)\varphi(\Phi^{-1}(\nu_1 J_0(t)))}{\Phi^{-1}(\nu_1 J_0(t))}\Big)^{\frac{1}{4}}.$$

Then, based on conditions (3.3), (3.5), (3.7) and taking into account that $\varphi'(\Phi^{-1}(z))$ is a regularly varying function of order -4 as $z\to Z_0$, we have

$$\begin{aligned}
\frac{q'(t)|H|^{\frac{1}{2}}(t)J_3(t)}{J_3'(t)} &\sim q'(t)\,\frac{\Phi^{-1}(\nu_1 J_0(t))}{(\Phi^{-1}(\nu_1 J_0(t)))'}\,\Big(\frac{\Phi^{-1}(\nu_1 J_0(t))\varphi'(\Phi^{-1}(\nu_1 J_0(t)))}{\varphi(\Phi^{-1}(\nu_1 J_0(t)))}\Big)^{\frac{1}{2}}\\
&= \nu_1\alpha_0 q'(t)\Big(\frac{\Phi^{-1}(\nu_1 J_0(t))}{p_0(t)\varphi(\Phi^{-1}(\nu_1 J_0(t)))}\Big)^{\frac{1}{4}}\Big(\frac{\Phi^{-1}(\nu_1 J_0(t))\varphi'(\Phi^{-1}(\nu_1 J_0(t)))}{\varphi(\Phi^{-1}(\nu_1 J_0(t)))}\Big)^{\frac{1}{2}}\\
&= \frac{\nu_1\alpha_0 q'(t)}{p_0^{\frac{1}{4}}}\Big(\frac{\Phi^{-1}(\nu_1 J_0(t))}{\varphi(\Phi^{-1}(\nu_1 J_0(t)))}\Big)^{\frac{3}{4}}\big(\varphi'(\Phi^{-1}(\nu_1 J_0(t)))\big)^{\frac{1}{2}} \quad \text{as } t\uparrow\omega.
\end{aligned}$$

Assuming that the limit as $t\to\omega$ of the expression on the left-hand side is a non-zero constant, from the above we obtain the following relation:

$$q'(t) = \nu_1\alpha_0 p_0^{\frac{1}{4}}(t)\Big(\frac{\Phi^{-1}(\nu_1 J_0(t))}{\varphi(\Phi^{-1}(\nu_1 J_0(t)))}\Big)^{-\frac{3}{4}}\big(\varphi'(\Phi^{-1}(\nu_1 J_0(t)))\big)^{-\frac{1}{2}}\xi(t),$$

where $\lim\limits_{t\to\omega}\xi(t)=\begin{cases}const\neq 0,\\ \pm\infty.\end{cases}$

Integrating this ratio over the interval from t_0 to t and taking into account the second conditions (3.2), (3.5), (3.8) and condition (3.20), we obtain

$$\int\limits_{t_0}^{\omega}\nu_1\alpha_0 p_0^{\frac{1}{4}}(t)\Big(\frac{\Phi^{-1}(\nu_1 J_0(t))}{\varphi(\Phi^{-1}(\nu_1 J_0(t)))}\Big)^{-\frac{3}{4}}\big(\varphi'(\Phi^{-1}(\nu_1 J_0(t)))\big)^{-\frac{1}{2}}\,dt$$
$$=\int\limits_{z(t_0)}^{Z_0}\Big(\frac{\Phi^{-1}(z)}{\varphi(\Phi^{-1}(z))}\Big)^{-\frac{3}{4}}\big(\varphi'(\Phi^{-1}(z))\big)^{-\frac{1}{2}}\,dz=\int\limits_{y(t_0)}^{Y_0}\Big(\frac{y\varphi'(y)}{\varphi(y)}\Big)^{-\frac{1}{2}}\frac{dy}{y}=\pm\infty,$$

So, $\lim\limits_{t\to\omega}q(t)=\pm\infty$.

However, this is impossible, since, according to (3.7), $q(t)\to 1$ as $t\to\omega$. Thus, taking into account the existence of the limit on the left-hand side of (3.19) and the condition (3.20), this limit can only be zero. □

Theorem 3.2. *Let $p_0:[a,\omega[\to]0,+\infty[$ be a continuously differentiable function, and along with (3.5)–(3.8) the conditions*

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2}\Big|\frac{y\varphi'(y)}{\varphi(y)}\Big|^{\frac{3}{4}}=0,\quad \lim_{t\uparrow\omega}\frac{q'(t)|H|^{\frac{1}{2}}(t)J_3(t)}{J_3'(t)}=0 \tag{3.21}$$

hold. Then, in the case where $\alpha_0\mu_0=-1$, the differential equation (1.1) has a two-parameter family of $P_\omega(Y_0,1)$-solutions, each satisfying the asymptotic representations

$$y'(t)=\alpha_0 J_3(t)\Big[1+\frac{o(1)}{|H(t)|^{\frac{3}{4}}}\Big],\quad y''(t)=\alpha_0 J_2(t)\Big[1+\frac{o(1)}{|H(t)|^{\frac{1}{2}}}\Big],\quad y'''(t)=\alpha_0 J_1(t)\Big[1+\frac{o(1)}{|H(t)|^{\frac{1}{4}}}\Big] \tag{3.22}$$

as $t\uparrow\omega$.

Proof. Let us assume that $p_0:[a,\omega[\to]0,+\infty[$ is a continuously differentiable function and conditions (3.21) are satisfied along with conditions (3.5)–(3.8). We prove that in this case, when $\alpha_0\mu_0=-1$, the differential equation (1.1) has at least one solution that allows asymptotic representations (3.22) as $t\uparrow\omega$, and we clarify the question of the number of such solutions. Since $p(t)\sim p_0(t)$ as $t\uparrow\omega$, the representation $p(t)=p_0(t)[1+r(t)]$ holds, where $r:[a,\omega[\to]-1,+\infty[$ is a continuous function such that $\lim\limits_{t\uparrow\omega}r(t)=0$. Taking this fact into account, the differential equation (1.1) can be transformed using the substitutions

$$y(t)=\Phi^{-1}(\nu_1 J_0(t))\Big[1+\frac{y_1(t)}{H(t)}\Big],\quad y^{(k)}(t)=\alpha_0 J_{4-k}(t)[1+y_{k+1}(t)]\ \ (k=1,2,3) \tag{3.23}$$

into a system of differential equations

$$\begin{cases}y_1'=E(t)\big[1-q(t)+h(t)q(t)y_1+y_2\big],\\[4pt] y_2'=\dfrac{J_3'(t)}{J_3(t)}\,[y_3-y_2],\\[4pt] y_3'=\dfrac{J_2'(t)}{J_2(t)}\,[y_4-y_3],\\[4pt] y_4'=\dfrac{J_1'(t)}{J_1(t)}\,\big[-1-y_4+G(t,y_1)[1+r(t)]\big],\end{cases} \tag{3.24}$$

where

$$E(t)=\alpha_0 J_3(t)\,\frac{\varphi'(\Phi^{-1}(\nu_1 J_0(t)))}{\varphi(\Phi^{-1}(\nu_1 J_0(t)))}\,,\quad h(t)=\frac{\big(\frac{\varphi'(z)}{\varphi(z)}\big)'}{\big(\frac{\varphi'(z)}{\varphi(z)}\big)^2}\bigg|_{z=\Phi^{-1}(\nu_1 J_0(t))}\,,$$

$$G(t,y_1)=\frac{\varphi(\Phi^{-1}(\nu_1 J_0(t))+\frac{\varphi(\Phi^{-1}(\nu_1 J_0(t)))}{\varphi'(\Phi^{-1}(\nu_1 J_0(t)))}y_1)}{\varphi(\Phi^{-1}(\nu_1 J_0(t)))}\,.$$

Taking into account condition (3.5), the first condition (3.8), condition (3.1) and the last condition (1.3), we choose a number $t_1\in[a,\omega[$ such that

$$\begin{gathered}\nu_1 J_0(t)\in\Delta_{Z_0}\ \text{ for }\ t\in[t_1,\omega[\,,\\ \Phi^{-1}(\nu_1 J_0(t))+\frac{\varphi(\Phi^{-1}(\nu_1 J_0(t)))}{\varphi'(\Phi^{-1}(\nu_1 J_0(t)))}\,y_1\in\Delta_{Y_0}\ \text{ for }\ t\in[t_1,\omega[\ \text{ and }\ |y_1|\le\frac{1}{4}\,.\end{gathered}\tag{3.25}$$

Let us consider this system of equations on the set

$$\Omega=[t_1,\omega[\,\times\mathbb{R}^4_{\frac{1}{4}}\,,\ \text{ where }\ \mathbb{R}^4_{\frac{1}{4}}=\Big\{(y_1,y_2,y_3,y_4):\ |y_i|\le\frac{1}{4}\ \ (i=1,\dots,4)\Big\}.$$

At the same time, we note that according to the first of conditions (3.1), (3.8), (3.15) and the second of conditions (1.3),

$$\lim_{t\uparrow\omega}\Phi^{-1}(\nu_1 J_0(t))=Y_0,\quad \lim_{t\uparrow\omega}H(t)=\pm\infty.\tag{3.26}$$

On the set $[t_1,\omega[\,\times\mathbb{R}^4_{\frac{1}{4}}$, the right-hand sides of the system of equations (3.24) are continuous, and the function G has continuous partial derivatives up to the second order inclusive with respect to the variable y_1. Expanding the function G for a fixed $t\in[t_1,\omega[$, by using Maclaurin's formula with a remainder term in the Lagrange form up to the second-order terms, we obtain

$$G(t,y_1)=1+y_1+R(t,y_1),\tag{3.27}$$

where

$$R(t,y_1)=\frac{\varphi(\Phi^{-1}(\nu_1 J_0(t)))\varphi''\big(\Phi^{-1}(\nu_1 J_0(t))+\frac{\varphi(\Phi^{-1}(\nu_1 J_0(t)))}{\varphi'(\Phi^{-1}(\nu_1 J_0(t)))}\,\xi_1\big)}{\varphi'^2(\Phi^{-1}(\nu_1 J_0(t)))}\,y_1^2,\ \ |\xi_1|<|y_1|.$$

From (3.22) and (3.23), together with the last condition in (1.3), it follows that

$$\varphi''\Big(\Phi^{-1}(\nu_1 J_0(t))+\frac{\varphi(\Phi^{-1}(\nu_1 J_0(t)))}{\varphi'(\Phi^{-1}(\nu_1 J_0(t)))}\,\xi_1\Big)=\frac{\varphi'^2\big(\Phi^{-1}(\nu_1 J_0(t))+\frac{\varphi(\Phi^{-1}(\nu_1 J_0(t)))}{\varphi'(\Phi^{-1}(\nu_1 J_0(t)))}\,\xi_1\big)}{\varphi\big(\Phi^{-1}(\nu_1 J_0(t))+\frac{\varphi(\Phi^{-1}(\nu_1 J_0(t)))}{\varphi'(\Phi^{-1}(\nu_1 J_0(t)))}\,\xi_1\big)}\,[1+r_1(t,y_1)],$$

where $\lim\limits_{t\uparrow\omega}r_1(t,y_1)=0$ uniformly for $y_1\in[-\frac{1}{4},\frac{1}{4}]$. By virtue of Lemma 2.3, the functions $\varphi(y),\varphi‘(y)\in\Gamma(Y_0,Z_0)$ with the auxiliary function given by $g(y)=\frac{-4\varphi(y)}{\varphi‘(y)}$. Therefore, using (2.2), (2.3) and Lemma 2.1, the last asymptotic relation can be written as

$$\varphi''\Big(\Phi^{-1}(\nu_1 J_0(t))+\frac{\varphi(\Phi^{-1}(\nu_1 J_0(t)))}{\varphi'(\Phi^{-1}(\nu_1 J_0(t)))}\,\xi_1\Big)=\frac{\varphi'^2(\Phi^{-1}(\nu_1 J_0(t)))}{\varphi(\Phi^{-1}(\nu_1 J_0(t)))}\,e^{-\frac{\xi}{4}}[1+r_2(t,y_1)],$$

where $\lim\limits_{t\uparrow\omega}r_2(t,y_1)=0$ uniformly for $y_1\in[-\frac{1}{4},\frac{1}{4}]$. It follows that $R(t,y_1)=e^{-\frac{\xi}{4}}[1+r_2(t,y_1)]y_1^2$, where $|\xi|<|y_1|$ and r_2 satisfies condition (3.25). Owing to this representation, for every $\varepsilon>0$, there exist $t_0\in[t_1,\omega[$ and $0<\delta\le\frac{1}{4}$ such that

$$|R(t,y_1)|\le(1+\varepsilon)y_1^2\ \text{ for }\ t\in[t_0,\omega[\,,\ \ |y_1|<\delta.\tag{3.28}$$

Hereinafter, we will assume that the number $\varepsilon > 0$ is chosen arbitrarily. Taking into account (3.27), we write the system of equations (3.24) in the form

$$\begin{cases} y_1' = E(t)[1 - q(t) + h(t)q(t)y_1 + y_2], \\ y_2' = \dfrac{J_3'(t)}{J_3(t)}\,[y_3 - y_2], \\ y_3' = \dfrac{J_2'(t)}{J_2(t)}\,[y_4 - y_3], \\ y_4' = \dfrac{J_1'(t)}{J_1(t)}\,\big[r(t) + y_1(1 + r(t)) - y_4 + R(t, y_1)(1 + r(t))\big] \end{cases} \tag{3.29}$$

and examine it on a set $[t_0, \omega[\, \times \mathbb{R}^4_\delta$, where $\mathbb{R}^4_\delta = (y_1, y_2, y_3, y_4) : \ |y_1| \le \delta, \ |y_2| \le \delta, \ \ |y_3| \le \delta, \ |y_4| \le \delta$. Using the transformation

$$y_1 = z_1, \quad y_k = q(t) - 1 + z_k \ \ (k = 2, 3, 4), \tag{3.30}$$

system (3.29) is reduced to a system of differential equations:

$$\begin{cases} z_1' = E(t)[h(t)q(t)z_1 + z_2], \\ z_2' = \dfrac{J_3'(t)}{J_3(t)}\Big[- q'(t)\,\dfrac{J_3(t)}{J_3'(t)} + z_3 - z_2\Big], \\ z_3' = \dfrac{J_2'(t)}{J_2(t)}\Big[- q'(t)\,\dfrac{J_2(t)}{J_2'(t)} + z_4 - z_3\Big], \\ z_4' = \dfrac{J_1'(t)}{J_1(t)}\Big[- q'(t)\,\dfrac{J_1(t)}{J_1'(t)} + r(t) + 1 - q(t) + (1 + r(t))z_1 - z_4 + R(t, z_1)(1 + r(t))\Big]. \end{cases} \tag{3.31}$$

In the system of equations (3.31), the factors on the right-hand side before the square brackets in the second, third, and fourth equations are equivalent as $t \uparrow \omega$ according to condition (3.7). As for the ratio of the factors in the first and second equations, based on (3.7) and the second condition in (3.26), we have

$$\frac{E(t)J_3(t)}{J_3'(t)} = \frac{\alpha_0 J_3(t)}{\Phi^{-1}(\nu_1 J_0(t))}\,\frac{J_3(t)}{J_3'(t)}\,H(t) \sim H(t) \to \pm\infty \ \text{ as } \ t \uparrow \omega.$$

To asymptotically equalise these coefficients, we apply an additional transformation to system (3.31)

$$z_1 = \upsilon_1, \quad z_2 = |H(t)|^{-\frac{3}{4}}\upsilon_2, \quad z_3 = |H(t)|^{-\frac{1}{2}}\upsilon_3, \quad z_4 = |H(t)|^{-\frac{1}{4}}\upsilon_4. \tag{3.32}$$

As a result, we obtain a system of differential equations of the form

$$\upsilon_i' = h(t)\Big[f_i(t) + \sum_{k=1}^{4} c_{ik}(t)\upsilon_k + V_i(t, \upsilon_1, \dots, \upsilon_i)\Big] \ \ (i = 1, \dots, 4), \tag{3.33}$$

where

$$h_1(t) = \frac{\alpha_0 J_3(t)|H(t)|^{\frac{1}{4}}}{\Phi^{-1}(\nu_1 J_0(t))}\,, \quad h_2(t) = \frac{J_3'(t)|H(t)|^{\frac{1}{4}}}{J_3(t)}\,,$$

$$h_3(t) = \frac{J_2'(t)|H(t)|^{\frac{1}{4}}}{J_2(t)}\,, \quad h_4(t) = \frac{J_1'(t)|H(t)|^{\frac{1}{4}}}{J_1(t)}\,,$$

$$f_1(t) = 0, \quad f_2(t) = -\frac{q'(t)J_3(t)|H(t)|^{\frac{1}{2}}}{J_3'(t)}\,,$$

$$f_3(t) = -\frac{q'(t)J_2(t)|H(t)|^{\frac{1}{4}}}{J_2'(t)}\,, \quad f_4(t) = -\frac{q'(t)J_1(t)}{J_1'(t)} + r(t) + 1 - q(t),$$

$$c_{11}(t) = q(t)h(t)H(t)^{\frac{3}{4}}, \quad c_{12}(t) \equiv \alpha_0\mu_0, \quad c_{13}(t) \equiv 0, \quad c_{14}(t) \equiv 0,$$
$$c_{21}(t) \equiv 0, \quad c_{22}(t) = \frac{3}{4}\,\frac{H'(t)J_3(t)}{|H(t)|^{\frac{5}{4}}J_3'(t)} - |H(t)|^{-\frac{1}{4}}, \quad c_{23}(t) \equiv 1, \quad c_{24}(t) \equiv 0,$$
$$c_{31}(t) \equiv 0, \quad c_{32}(t) \equiv 0, \quad c_{33}(t) = \frac{1}{2}\,\frac{H'(t)J_2(t)}{|H(t)|^{\frac{5}{4}}J_2'(t)} - |H(t)|^{-\frac{1}{4}}, \quad c_{34}(t) \equiv 1.$$
$$c_{41}(t) = 1 + r(t), \quad c_{42}(t) \equiv 0, \quad c_{43}(t) \equiv 0, \quad c_{44}(t) = \frac{1}{4}\,\frac{H'(t)J_1(t)}{|H(t)|^{\frac{5}{4}}J_1'(t)} - |H(t)|^{-\frac{1}{4}},$$
$$V_i(t, v_1, v_2, v_3, v_4) \equiv 0, \;\; i = 1, 2, 3, \quad V_4(t, v_1, v_2, v_3, v_4) = R(t, v_1).$$

Here, the functions $h_i(t)$ satisfy the following conditions:

$$h_i(t) \neq 0 \;\text{ and for }\; t_0 \le t < \omega, \quad \int_{t_0}^{\omega} h_i(t)\,dt = \pm\infty.$$

By virtue of the second condition (3.21), conditions (3.8) and the introduced notations (3.1), we obtain

$$\lim_{t\uparrow\omega} f_i(t) = 0, \;\; i = 1, 2, 3, 4 \;\text{ as }\; t \uparrow \omega,$$
$$\lim_{t\uparrow\omega} c_{11}(t) = 0, \quad \lim_{t\uparrow\omega} c_{12}(t) = \alpha_0\mu_0, \quad \lim_{t\uparrow\omega} c_{13}(t) = 0, \quad \lim_{t\uparrow\omega} c_{14}(t) = 0,$$
$$\lim_{t\uparrow\omega} c_{21}(t) = 0, \quad \lim_{t\uparrow\omega} c_{23}(t) = 1, \quad \lim_{t\uparrow\omega} c_{24}(t) = 0,$$
$$\lim_{t\uparrow\omega} c_{31}(t) = 0, \quad \lim_{t\uparrow\omega} c_{32}(t) = 0, \quad \lim_{t\uparrow\omega} c_{34}(t) = 1,$$
$$\lim_{t\uparrow\omega} c_{41}(t) = 1, \quad \lim_{t\uparrow\omega} c_{42}(t) = 0, \quad \lim_{t\uparrow\omega} c_{43}(t) = 0.$$

Furthermore, considering that

$$\frac{H'(t)}{|H(t)|^{\frac{5}{4}}} = \frac{(\Phi^{-1}(\nu_1 J_0(t)))'}{\Phi(\nu_1 J_0(t))}\left[|H(t)|^{-\frac{1}{4}} + h(t)|H(t)|^{\frac{3}{4}}\right]$$

and the first condition (3.21) is satisfied, using (3.7), we obtain

$$\lim_{t\uparrow\omega} \frac{H'(t)J_1(t)}{|H(t)|^{\frac{5}{4}}(t)J_1'(t)} = 0, \quad \lim_{t\uparrow\omega} \frac{H'(t)J_2(t)}{|H(t)|^{\frac{5}{4}}(t)J_2'(t)} = 0, \quad \lim_{t\uparrow\omega} \frac{H'(t)J_3(t)}{|H(t)|^{\frac{5}{4}}(t)J_3'(t)} = 0,$$

that is,

$$\lim_{t\uparrow\omega} c_{22}(t) = 0, \quad \lim_{t\uparrow\omega} c_{33}(t) = 0, \quad \lim_{t\uparrow\omega} c_{44}(t) = 0.$$

Considering the above, the boundary matrix of the coefficients $C(c_{ik}(t))_{i,k=1}^4$ of system (3.33) as $t \uparrow \omega$ has the following form:

$$C = \begin{pmatrix} 0 & \alpha_0\mu_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.34}$$

The characteristic equation of this matrix is

$$\lambda^4 = \alpha_0\mu_0. \tag{3.35}$$

In the case where $\alpha_0\mu_0 = -1$, equation (3.35) has two roots with a positive real part and two roots with a negative real part. That is,

$$\lambda_{k+1} = e^{i\,\frac{\pi + 2\pi k}{4}} = \cos\Big(\frac{\pi + 2\pi k}{4}\Big) + i\sin\Big(\frac{\pi + 2\pi k}{4}\Big), \;\; k = 0, 1, 2, 3,$$

$$\lambda_{1,4} = \frac{\sqrt{2}}{2}(1 \pm i), \quad \lambda_{2,3} = \frac{\sqrt{2}}{2}(-1 \pm i).$$

Finally, note that according to the estimate (3.28),

$$\lim_{\upsilon_1 \to 0} \frac{V(t, \upsilon_1)}{\upsilon_1} \text{ evenly across } t \in [t_0, \omega[\,.$$

Thus, for the system of differential equations (3.33), all the conditions of Theorem 2.2 from [6] are satisfied. Based on this theorem, system (3.33) has a two-parameter family of solutions $(\upsilon_i)_{i=1}^4 : [t_0, \omega[\to \mathbb{R}^4$ $(t_1 \in [t_0, \omega])$ tending to zero as $t \uparrow \omega$. According to transformations (3.30) and (3.32), each such solution of the system corresponds to the solution $y : [t_1, \omega[$ of the differential equation (1.1) satisfying the asymptotic representations (3.22) as $t \uparrow \omega$. It is also easy to verify that each of these solutions is a $P_\omega(Y_0, \lambda_0)$-solution of the differential equation (1.1). □

Comment. The question of the actual existence of $P_\omega(Y_0, \lambda_0)$-solutions of differential equation (1.1) admitting asymptotic representations (3.8)–(3.10) as $t \uparrow \omega$ in the case $\alpha_0 \mu_0 = 1$ remains open.

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