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Aurelian Cernea

ON AN ANTI-PERIODIC BOUNDARY VALUE PROBLEM FOR A CAPUTO–FABRIZIO FRACTIONAL DIFFERENTIAL INCLUSION

Abstract. The existence of solutions for a Caputo–Fabrizio fractional differential inclusion with anti-periodic boundary conditions is investigated. New results are obtained when the right hand side has convex or non convex values.

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1 Introduction

In the last years one may see a strong development of the theory of differential equations and inclusions of fractional order. The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena. In the fractional calculus, there are several fractional derivatives. From them, the fractional derivative introduced by Caputo in [7] allows to use Cauchy conditions which have physical meanings.

Recently, a new fractional order derivative with a regular kernel has been introduced by Caputo and Fabrizio [8]. The Caputo–Fabrizio operator is useful for modeling several classes of problems with the dynamics having the exponential decay law. This new definition is able to describe better heterogeneousness (heterogeneity?), systems with different scales with memory effects, the wave movement on a surface of shallow water, the heat transfer model, mass-spring-damper model. Another good property of this new definition is that, using the Laplace transform of the fractional derivative, the fractional differential equation turns into a classical differential equation of integer order. The properties of this definition have been studied in [1,2,8,9]. Several recent papers are devoted to qualitative results for fractional differential equations and inclusions defined by the Caputo–Fabrizio fractional derivative (see [18–20], etc.).

The present paper is devoted to the following boundary value problem:

$$D^{\rho}_{CF}x(t) + \lambda x(t) \in F(t, x(t)) \ \text{ a.e. } \ ([0,T]), \quad x(0) = -x(T), \tag{1.1}$$

where $F(\,\cdot\,,\,\cdot\,) : [0,T] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map, $\mathcal{P}(\mathbf{R})$ is a family of all nonempty subsets of $\mathbf{R}$, $\lambda > 0$ and D^{ρ}_{CF} denotes a Caputo–Fabrizio's fractional derivative of order $\rho \in (0,1)$.

Our study is motivated by the recent paper [5], where problem (1.1) is studied in the situation when $F(\,\cdot\,,\,\cdot\,)$ is single valued and the existence of solutions is obtained by using a nonlinear alternative of Leray–Schauder type and by a monotone iterative method of coupled lower and upper solutions.

The aim of our paper is to consider the set-valued problems in a more general framework and to present three existence results for problem (1.1). Our results are obtained under several hypotheses concerning the regularity of the set-valued map F and are based on a nonlinear alternative of Leray–Schauder type, on Bressan–Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values and on Kuratowski and Ryll–Nardzewski selection theorem. We underline that the methods used are quite well known in the theory of differential inclusions, however, their exposition in the framework of problem (1.1) is new.

Also, we mention that similar results for an anti-periodic boundary problem associated with a fractional differential inclusion defined by the original Caputo fractional derivative are obtained in [10], for a Cauchy and for a bilocal problem associated with the Caputo–Fabrizio fractional differential inclusion can be found in our previous papers [11] and [12].

The paper is organized as follows. In Section 2, we recall some preliminary facts that will be needed in the sequel. In Section 3, we prove our results by using fixed point techniques. In Section 4, we provide a Filippov type existence result.

2 Preliminaries

In this section, we sum up some basic facts that we are going to use later. Let (X,d) be a metric space with the corresponding norm $|\cdot|$ and denote $I = [0,T]$. Denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$, then $\chi_A(\,\cdot\,) : I \to \{0,1\}$ denotes the characteristic function of A. For any subset $A \subset X$, we denote by $\overline{A}$ the closure of A. Recall that the Pompeiu–Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\big\{d^*(A,B), d^*(B,A)\big\},$$

where

$$d^*(A,B) = \sup\big\{d(a,B) :\ a \in A\big\} \ \text{ and } \ d(x,B) = \inf_{y\in B} d(x,y).$$

As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x(\cdot):I\to X$ endowed with the norm
$$|x(\cdot)|_C=\sup_{t\in I}|x(t)|,$$
by $AC(I,X)$ the Banach space of all absolutely continuous functions $x(\cdot):I\to X$ and by $L^p(I,X)$ the Banach space of all (Bochner) p-integrable functions $x(\cdot):I\to X$; in particular, $L^1(I,X)$ is the Banach space of all (Bochner) integrable functions $x(\cdot):I\to X$ endowed with the norm
$$|x(\cdot)|_1=\int\limits_I|x(t)|\,dt.$$
A subset $D\subset L^1(I,X)$ is said to be *decomposable* if for any $u(\cdot),v(\cdot)\in D$ and any subset $A\in\mathcal{L}(I)$ one has $u\chi_A+v\chi_B\in D$, where $B=I\setminus A$.

Consider $M:X\to\mathcal{P}(X)$ a set-valued map. A point $x\in X$ is called a fixed point for $M(\cdot)$ if $x\in M(x)$. $M(\cdot)$ is said to be bounded on bounded sets if $M(B):=\bigcup\limits_{x\in B}M(x)$ is a bounded subset of X for all bounded sets B in X. $M(\cdot)$ is said to be compact if $\overline{M(B)}$ is relatively compact for any bounded sets B in X. $M(\cdot)$ is said to be totally compact if $\overline{M(X)}$ is a compact subset of X. $M(\cdot)$ is said to be upper semicontinuous if for any $x_0\in X$, $M(x_0)$ is a nonempty closed subset of X and if for each open set D of X containing $M(x_0)$ there exists an open neighborhood V_0 of x_0 such that $M(V_0)\subset D$. Let E be a Banach space, $Y\subset E$ be a nonempty closed subset and $M(\cdot):Y\to\mathcal{P}(E)$ be a multifunction with nonempty closed values. $M(\cdot)$ is said to be lower semicontinuous if for any open subset $D\subset E$, the set $\{y\in Y:\ M(y)\cap D\neq\varnothing\}$ is open. $M(\cdot)$ is called completely continuous if it is upper semicontinuous and totally compact on X. It is well known that a compact set-valued map $M(\cdot)$ with nonempty compact values is upper semicontinuous if and only if $M(\cdot)$ has a closed graph.

The next results are the key tools in the proof of our theorems. We recall, first, the following Leray–Schauder type nonlinear alternative proved in [17] and its consequences.

Theorem 2.1. *Let D and $\overline{D}$ be the open and closed subsets in a normed linear space X such that $0\in D$, and let $M:\overline{D}\to\mathcal{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

(i) *the inclusion $x\in M(x)$ has a solution, or*

(ii) *there exists $x\in\partial D$ (the boundary of D) such that $\lambda x\in M(x)$ for some $\lambda>1$.*

Corollary 2.1. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r, and let $M:\overline{B_r(0)}\to\mathcal{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

(i) *the inclusion $x\in M(x)$ has a solution, or*

(ii) *there exists $x\in X$ with $|x|=r$ and $\lambda x\in M(x)$ for some $\lambda>1$.*

Corollary 2.2. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r, and let $M:\overline{B_r(0)}\to X$ be a completely continuous single valued map with compact convex values. Then either*

(i) *the equation $x=M(x)$ has a solution, or*

(ii) *there exists $x\in X$ with $|x|=r$ and $x=\lambda M(x)$ for some $\lambda<1$.*

If $G(\cdot,\cdot):I\times X\to\mathcal{P}(X)$ is a set-valued map with compact values, we define $S_G:C(I,X)\to\mathcal{P}(L^1(I,X))$ by
$$S_G(x):=\Big\{g\in L^1(I,X):\ g(t)\in G(t,x(t))\ \text{ a.e. } (I)\Big\}.$$
We say that $G(\cdot,\cdot)$ is of *lower semicontinuous type* if $S_G(\cdot)$ is lower semicontinuous with nonempty closed and decomposable values. The next result is proved in [6].

Theorem 2.2. *Let S be a separable metric space and $G(\cdot): S \to \mathcal{P}(L^1(I,X))$ be a lower semicontinuous set-valued map with closed decomposable values.*

Then $G(\cdot)$ has a continuous selection (i.e., there exists a continuous mapping $g(\cdot): S \to L^1(I,X)$ such that $g(s) \in G(s)$ for all $s \in S$).

A set-valued map $G: I \to \mathcal{P}(X)$ with nonempty compact convex values is said to be *measurable* if for any $x \in X$, the function $t \to d(x, G(t))$ is measurable. A set-valued map $G(\cdot,\cdot): I \times X \to \mathcal{P}(X)$ is said to be *Carathéodory* if $t \to G(t,x)$ is measurable for any $x \in X$ and $x \to G(t,x)$ is upper semicontinuous for almost all $t \in I$. Moreover, $G(\cdot,\cdot)$ is said to be L^1-*Carathéodory* if for any $r > 0$, there exists $p_r(\cdot) \in L^1(I, \mathbf{R})$ such that $\sup\{|v| : \ v \in G(t,x)\} \leq p_r(t)$ a.e. (I), $\forall x \in \overline{B_r(0)}$. The following theorem is proved in [16].

Theorem 2.3. *Let X be a Banach space, let $G(\cdot,\cdot): I \times X \to \mathcal{P}(X)$ be an L^1-Carathéodory set-valued map with $S_G(x) \neq \varnothing$ for all $x(\cdot) \in C(I,X)$ and let $\Gamma : L^1(I,X) \to C(I,X)$ be a linear continuous mapping.*

Then the set-valued map $\Gamma \circ S_G : C(I,X) \to \mathcal{P}(C(I,X))$ defined by

$$(\Gamma \circ S_G)(x) = \Gamma(S_G(x))$$

has compact convex values and has a closed graph in $C(I,X) \times C(I,X)$.

Note that if $dim X < \infty$, and $G(\cdot,\cdot)$ is as in Theorem 2.3, then $S_G(x) \neq \varnothing$ for any $x(\cdot) \in C(I,X)$ (see, e.g., [16]).

We recall also a selection result in [3] which is a version of the celebrated Kuratowski and Ryll–Nardzewski selection theorem.

Lemma 2.1. *Consider X is a separable Banach space, B is the closed unit ball in X, $H: I \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g: I \to X$, $L: I \to \mathbf{R}_+$ are measurable functions. If*

$$H(t) \cap (g(t) + L(t)B) \neq \varnothing \ \ a.e. \ (I),$$

then the set-valued map $t \to H(t) \cap (g(t) + L(t)B)$ has a measurable selection.

The next definitions were introduced by Caputo and Fabrizio in [8].

Definition 2.1.

(a) The Caputo–Fabrizio integral of order $\rho \in (0,1)$ of a function $f \in L^1(I, \mathbf{R})$ is defined by

$$I^{\rho}_{CF} f(t) = \frac{2(1-\rho)}{M(\rho)(2-\rho)} f(t) + \frac{2\rho}{M(\rho)(2-\rho)} \int_0^t f(s)\, ds,$$

where $M(\rho)$ is a normalization constant depending on ρ.

(b) The Caputo–Fabrizio fractional derivative of order $\rho \in (0,1)$ of a function $f \in AC(I, \mathbf{R})$ is defined by

$$D^{\rho}_{CF} f(t) = \frac{M(\rho)(2-\rho)}{2(1-\rho)} \int_0^t e^{-\frac{\rho}{1-\rho}(t-s)} f'(s)\, ds.$$

Definition 2.2. A mapping $x(\cdot) \in AC(I, \mathbf{R})$ is called a *solution* of problem (1.1) if there exists a function $f(\cdot) \in L^1(I, \mathbf{R})$ such that

$$\begin{gathered} f(t) \in F(t, x(t)) \ \text{ a.e. } (I), \\ D^{\rho}_{CF} x(t) = f(t), \quad t \in I, \quad x(0) = -x(T). \end{gathered}$$

In order to prove our results, we also need the following result proved in [5] (namely, Lemma 2).

Lemma 2.2. *If $f(\cdot) \in L^1(I, \mathbf{R})$, then the problem*

$$D^{\rho}_{CF}x(t) + \lambda x(t) = f(t) \ \ a.e.\ (I), \quad x(0) = -x(T),$$

has a unique solution given by $x(t) = \int\limits_0^T G(t,s)f(s)\,ds$, where

$$G(t,s) := \begin{cases} \dfrac{e^{\frac{\lambda\rho}{1+\lambda(1-\rho)}(T-t+s)}}{e^{\frac{\lambda\rho}{1+\lambda(1-\rho)}T}+1} & \text{if } 0 \le s < t \le T, \\ -\dfrac{e^{\frac{\lambda\rho}{1+\lambda(1-\rho)}(s-t)}}{e^{\frac{\lambda\rho}{1+\lambda(1-\rho)}T}+1} & \text{if } 0 \le t < s \le T. \end{cases}$$

Note that for any $s, t \in I$, $|G(t,s)| \le \frac{e^{2\lambda T}}{2}$.

3 Existence via fixed points

We now present the existence results for problem (1.1). Let us first the case when $F(\cdot,\cdot)$ is convex valued and upper semicontinuous in the state variable.

Hypothesis H1.

(i) $F(\cdot,\cdot) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty compact convex values and is Carathéodory.

(ii) There exist $\varphi(\cdot) \in L^1(I, \mathbf{R})$ with $\varphi(t) > 0$ a.e. (I) and a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\sup\left\{|v| : \ v \in F(t,x)\right\} \le \varphi(t)\psi(|x|) \ \text{ a.e. } (I), \ \ \forall\, x \in \mathbf{R}.$$

Theorem 3.1. *Assume that Hypothesis H1 is satisfied and there exists $r > 0$ such that*

$$r > \frac{e^{2\lambda T}}{2}\,|\varphi|_1\psi(r). \tag{3.1}$$

Then problem (1.1) *has at least one solution $x(\cdot)$ such that $|x(\cdot)|_C < r$.*

Proof. Let $X = C(I, \mathbf{R})$ and consider $r > 0$ as in (3.1). It is obvious that the existence of solutions to problem (1.1) reduces to the existence of solutions of the integral inclusion

$$x(t) \in \int\limits_0^T G(t,s)F(s,x(s))\,ds, \ \ t \in I.$$

Consider the set-valued map $S : \overline{B_r(0)} \to \mathcal{P}(C(I,\mathbf{R}))$ defined by

$$S(x) := \left\{ v(\cdot) \in C(I,\mathbf{R}) : \ \ v(t) := \int\limits_0^T G(t,s)f(s)\,ds, \ \ f \in \overline{S_F(x)} \right\}.$$

We show that $S(\cdot)$ satisfies the hypotheses of Corollary 2.1. First, we show that $S(x) \subset C(I,\mathbf{R})$ is convex for any $x \in C(I,\mathbf{R})$.

If $v_i \in S(x)$, then there exists $f_i \in S_F(x)$ such that for any $t \in I$ one has

$$v_i(t) = \int\limits_0^T G(t,s)f_i(s)\,ds, \ \ i = 1,2.$$

Let $0 \le \alpha \le 1$. Then for any $t \in I$ we have

$$(\alpha v_1 + (1-\alpha)v_2)(t) = \int_0^T G(t,s)\big[\alpha f_1(s) + (1-\alpha) f_2(s)\big]\, ds.$$

The values of $F(\,\cdot\,,\,\cdot\,)$ are convex, thus $S_F(x)$ is a convex set and hence $\alpha f_1 + (1-\alpha) f_2 \in S(x)$.

Secondly, we show that $S(\,\cdot\,)$ is bounded on the bounded sets of $C(I,\mathbf{R})$.

Let $B \subset C(I,\mathbf{R})$ be a bounded set. Then there exists $m > 0$ such that $|x|_C \le m$, $\forall\, x \in B$. If $v \in T(x)$, there exists $f \in S_F(x)$ such that $v(t) = \int_0^T G(t,s) f(s)\, ds$. For any $t \in I$, one can write

$$|v(t)| \le \int_0^T |G(t,s)| \cdot |f(s)|\, ds \le \int_0^T |G(t,s)| \varphi(s) \psi(|x(t)|)\, ds$$

and therefore

$$|v|_C \le \frac{e^{2\lambda T}}{2}\, |\varphi|_1 \psi(m), \ \ \forall\, v \in S(x),$$

i.e., $S(B)$ is bounded.

We show next that $S(\,\cdot\,)$ maps the bounded sets into the equi-continuous sets.

Let $B \subset C(I,\mathbf{R})$ be a bounded set as before and $v \in S(x)$ for some $x \in B$. There exists $f \in S_F(x)$ such that $v(t) = \int_0^T G(t,s) f(s)\, ds$. Then for any $t, \tau \in I$, we have

$$\begin{aligned}|v(t) - v(\tau)| &\le \left| \int_0^T G(t,s) f(s)\, ds - \int_0^T G(\tau,s) f(s)\, ds \right| \\ &\le \int_0^T |G(t,s) - G(\tau,s)| \cdot |f(s)|\, ds \le \int_0^T |G(t,s) - G(\tau,s)| \varphi(s) \psi(m)\, ds.\end{aligned}$$

It follows that $|v(t) - v(\tau)| \to 0$ as $t \to \tau$. Therefore, $S(B)$ is an equi-continuous set in $C(I,\mathbf{R})$.

We apply now Arzela–Ascoli's theorem and deduce that $S(\,\cdot\,)$ is completely continuous on $C(I,\mathbf{R})$.

In the next step of the proof, we prove that $S(\,\cdot\,)$ has a closed graph.

Let $x_n \in C(I,\mathbf{R})$ be a sequence such that $x_n \to x^*$ and $v_n \in S(x_n)$, $\forall\, n \in \mathbf{N}$ such that $v_n \to v^*$. We prove that $v^* \in S(x^*)$. Since $v_n \in S(x_n)$, there exists $f_n \in S_F(x_n)$ such that $v_n(t) = \int_0^T G(t,s) f_n(s)\, ds$. Define $\Gamma : L^1(I,\mathbf{R}) \to C(I,\mathbf{R})$ by

$$(\Gamma(f))(t) := \int_0^T G(t,s) f(s)\, ds.$$

One has

$$\max_{t \in I} \big| v_n(t) - v^*(t) \big| = \big| v_n(\,\cdot\,) - v^*(\,\cdot\,) \big|_C \to 0 \ \text{ as } \ n \to \infty.$$

We apply Theorem 2.3 to find that $\Gamma \circ S_F$ has a closed graph and from the definition of Γ, we get $v_n \in \Gamma \circ S_F(x_n)$. Since $x_n \to x^*$, $v_n \to v^*$, it follows the existence of $f^* \in S_F(x^*)$ such that

$$v^*(t) = \int_0^T G(t,s) f^*(s)\, ds.$$

Therefore, $S(\,\cdot\,)$ is upper semicontinuous and compact on $\overline{B_r(0)}$. We apply Corollary 2.1 to deduce that either (i) the inclusion $x \in S(x)$ has a solution in $\overline{B_r(0)}$, or (ii) there exists $x \in X$ with $|x|_C = r$

and $\lambda x \in S(x)$ for some $\lambda > 1$. Assume that (ii) is true. With the same arguments as in the second step of our proof, we get

$$r = |x(\cdot)|_C \le \frac{e^{2\lambda T}}{2} |\varphi|_1 \psi(r)$$

which contradicts (3.1). Hence only (i) is valid and theorem is proved. □

We consider now the case when $F(\cdot,\cdot)$ is not necessarily convex valued. Our existence result in this case is based on the Leray–Schauder alternative for single valued maps and on Bressan Colombo selection theorem.

Hypothesis H2.

(i) $F(\cdot,\cdot) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has compact values, $F(\cdot,\cdot)$ is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ measurable and $x \to F(t,x)$ is lower semicontinuous for almost all $t \in I$.

(ii) There exist $\varphi(\cdot) \in L^1(I,\mathbf{R})$ with $\varphi(t) > 0$ a.e. (I) and a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\sup\big\{|v| : \ v \in F(t,x)\big\} \le \varphi(t)\psi(|x|) \ \text{ a.e. } (I), \ \ \forall\, x \in \mathbf{R}.$$

Theorem 3.2. *Assume that Hypothesis* H2 *is satisfied and there exists* $r > 0$ *such that condition* (3.1) *is satisfied.*

Then problem (1.1) *has at least one solution on* I*.*

Proof. We note first that if Hypothesis H2 is satisfied, then $F(\cdot,\cdot)$ is of lower semicontinuous type (e.g., [15]). Therefore, we apply Theorem 2.2 with $S = C(I,\mathbf{R})$ and $G(\cdot) = S_F(\cdot)$ to deduce that there exists a continuous mapping $f(\cdot) : C(I,\mathbf{R}) \to L^1(I,\mathbf{R})$ such that $f(x) \in S_F(x)$, $\forall\, x \in C(I,\mathbf{R})$. We consider the corresponding problem

$$x(t) = \int_0^T G(t,s) f(x(s))\, ds, \ \ t \in I, \tag{3.2}$$

in the space $X = C(I,\mathbf{R})$. It is clear that if $x(\cdot) \in C(I,\mathbf{R})$ is a solution of problem (3.2), then $x(\cdot)$ is a solution to problem (1.1).

Let $r > 0$ satisfy condition (3.1) and define the set-valued map $U : \overline{B_r(0)} \to \mathcal{P}(C(I,\mathbf{R}))$ by

$$(U(x))(t) := \int_0^T G(t,s) f(x(s))\, ds.$$

Obviously, the integral equation (3.2) is equivalent to the operator equation

$$x(t) = (U(x))(t), \ \ t \in I.$$

It remains to show that $U(\cdot)$ satisfies the hypotheses of Corollary 2.2.

We show that $U(\cdot)$ is continuous on $\overline{B_r(0)}$. From Hypotheses H2 (ii), we have

$$|f(x(t))| \le \varphi(t)\psi(|x(t)|) \ \text{ a.e. } (I)$$

for all $x(\cdot) \in C(I,\mathbf{R})$. Let x_n, $x \in \overline{B_r(0)}$ such that $x_n \to x$. Then

$$|f(x_n(t))| \le \varphi(t)\psi(r) \ \text{ a.e. } (I).$$

From Lebesgue's dominated convergence theorem and the continuity of $f(\cdot)$, we find that for all $t \in I$,

$$\lim_{n\to\infty} (U(x_n))(t) = \int_0^T G(t,s) f(x_n(s))\, ds = \int_0^T G(t,s) f(x(s))\, ds = (U(x))(t)$$

i.e., $U(\,\cdot\,)$ is continuous on $\overline{B_r(0)}$.

Repeating the arguments in the proof of Theorem 3.1 with the corresponding modifications, it follows that $U(\,\cdot\,)$ is compact on $\overline{B_r(0)}$. We apply Corollary 2.2 and find that either (i) the equation $x = U(x)$ has a solution in $\overline{B_r(0)}$, or (ii) there exists $x \in X$ with $|x|_C = r$ and $x = \lambda U(x)$ for some $\lambda < 1$.

As in the proof of Theorem 3.1, if the statement (ii) holds true, then we obtain a contradiction to (3.1). Thus only the statement (i) is true and problem (1.1) has a solution $x(\,\cdot\,) \in C(I, \mathbf{R})$ with $|x(\,\cdot\,)|_C < r$. □

4 A Filippov type existence result

In this section, we consider the even more general problem

$$D^{\rho}_{CF}x(t) + \lambda x(t) \in F(t, x(t), V(x)(t)) \text{ a.e. } ([0,T]), \quad x(0) = -x(T), \tag{4.1}$$

where $F : [0,T] \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map, $V : C([0,T], \mathbf{R}) \to C([0,T], \mathbf{R})$ is a nonlinear Volterra integral operator defined by $V(x)(t) = \int\limits_0^t k(t,s,x(s))\,ds$ with a given function $k(\,\cdot\,,\,\cdot\,,\,\cdot\,) : [0,T] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$. We show that Filippov's ideas in [14] may be suitably adapted in order to obtain the existence of solutions to problem (4.1).

Hypothesis H3.

(i) $F(\,\cdot\,,\,\cdot\,) : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \times \mathbf{R})$ measurable.

(ii) There exists $L(\,\cdot\,) \in L^1(I, (0,\infty))$ such that for almost all $t \in I$, $F(t,\,\cdot\,,\,\cdot\,)$ is $L(t)$-Lipschitz in the sense that

$$d_H\big(F(t,x_1,y_1), F(t,x_2,y_2)\big) \le L(t)\big(|x_1 - x_2| + |y_1 - y_2|\big), \quad \forall\, x_1, x_2, y_1, y_2 \in \mathbf{R}.$$

(iii) $k(\,\cdot\,,\,\cdot\,,\,\cdot\,) : I \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a function such that $\forall\, x \in \mathbf{R}$, $(t,s) \to k(t,s,x)$ is measurable.

(iv) $|k(t,s,x) - k(t,s,y)| \le L(t)|x - y|$ a.e. $(t,s) \in I \times I$, $\forall\, x, y \in \mathbf{R}$.

Below we use the following notation:

$$M(t) := L(t)\bigg(1 + \int\limits_0^t L(u)\,du\bigg), \quad t \in I, \quad M_0 = \int\limits_0^T M(t)\,dt.$$

Theorem 4.1. *Assume that Hypothesis H3 is satisfied and $e^{2\lambda T} M_0 < 2$. Let $y(\,\cdot\,) \in C(I, \mathbf{R})$ be such that $y(0) = -y(T)$ and there exists $p(\,\cdot\,) \in L^1(I, \mathbf{R}_+)$ with*

$$d\Big(D^{\rho}_{CF}y(t) + \lambda y(t), F\big(t, y(t), V(y)(t)\big)\Big) \le p(t) \text{ a.e. } (I).$$

Then there exists a solution $x(\,\cdot\,)$ of problem (4.1) satisfying

$$|x(t) - y(t)| \le \frac{e^{2\lambda T}}{2 - M_0 e^{2\lambda T}} \int\limits_0^T p(t)\,dt \text{ for all } t \in I.$$

Proof. The set-valued map $t \to F(t, y(t), V(y)(t))$ is measurable with closed values and

$$F(t, y(t), V(y)(t)) \cap \Big\{D^{\rho}_{CF}y(t) + \lambda y(t) + p(t)[-1,1]\Big\} \ne \varnothing \text{ a.e. } (I).$$

It follows from Lemma 2.1 that there exists a measurable selection $f_1(t) \in F(t, y(t), V(y)(t))$ a.e. (I) such that

$$\big|f_1(t) - D^{\rho}_{CF}y(t) - \lambda y(t)\big| \le p(t) \text{ a.e. } (I). \tag{4.2}$$

Define $x_1(t)=\int\limits_0^T G(t,s)f_1(s)\,ds$ and one has

$$|x_1(t)-y(t)|\leq \frac{e^{2\lambda T}}{2}\int\limits_0^T p(t)\,dt.$$

We claim that it is sufficient to construct the sequences $x_n(\cdot)\in C(I,\mathbf{R})$, $f_n(\cdot)\in L^1(I,\mathbf{R})$, $n\geq 1$, with the following properties:

$$x_n(t)=\int\limits_0^T G(t,s)f_n(s)\,ds,\quad t\in I, \tag{4.3}$$

$$f_n(t)\in F\big(t,x_{n-1}(t),V(x_{n-1})(t)\big)\ \text{ a.e. } (I), \tag{4.4}$$

$$|f_{n+1}(t)-f_n(t)|\leq L(t)\bigg(|x_n(t)-x_{n-1}(t)|+\int\limits_0^t L(s)|x_n(s)-x_{n-1}(s)|\,ds\bigg)\ \text{ a.e. } (I). \tag{4.5}$$

If this construction is realized, then from (4.2)–(4.5) for almost all $t\in I$, we have

$$|x_{n+1}(t)-x_n(t)|\leq \frac{e^{2\lambda T}}{2}\Big(\frac{e^{2\lambda T}M_0}{2}\Big)^n\int\limits_0^T p(t)\,dt,\quad \forall\, n\in\mathbf{N}.$$

Indeed, assume that the last inequality is true for $n-1$ and prove it for n. One has

$$\begin{aligned}
|x_{n+1}(t)-x_n(t)| &\leq \int\limits_0^T |G(t,t_1)|\cdot\big|f_{n+1}(t_1)-f_n(t_1)\big|\,dt_1\\
&\leq \frac{e^{2\lambda T}}{2}\int\limits_0^T L(t_1)\bigg[\big|x_n(t_1)-x_{n-1}(t_1)\big|+\int\limits_0^{t_1} L(s)|x_n(s)-x_{n-1}(s)|\,ds\bigg]\,dt_1\\
&\leq \frac{e^{2\lambda T}}{2}\int\limits_0^T L(t_1)\bigg(1+\int\limits_0^{t_1} L(s)\,ds\bigg)\,dt_1\cdot\Big(\frac{e^{2\lambda T}}{2}\Big)^n M_0^{n-1}\int\limits_0^T p(t)\,dt\\
&= \frac{e^{2\lambda T}}{2}\Big(\frac{e^{2\lambda T}M_0}{2}\Big)^n\int\limits_0^T p(t)\,dt.
\end{aligned}$$

Therefore, $\{x_n(\cdot)\}$ is a Cauchy sequence in the Banach space $C(I,\mathbf{R})$ converging uniformly to some $x(\cdot)\in C(I,\mathbf{R})$. Thus, by (4.5), for almost all $t\in I$, the sequence $\{f_n(t)\}$ is the Cauchy one in $\mathbf{R}$. Let $f(\cdot)$ be the pointwise limit of $f_n(\cdot)$. Moreover, one has

$$\begin{aligned}
|x_n(t)-y(t)| &\leq |x_1(t)-y(t)|+\sum_{i=1}^{n-1}|x_{i+1}(t)-x_i(t)|\\
&\leq \frac{e^{2\lambda T}}{2}\int\limits_0^T p(t)\,dt+\sum_{i=1}^{n-1}\bigg(\frac{e^{2\lambda T}}{2}\int\limits_0^T p(t)\,dt\bigg)\Big(\frac{e^{2\lambda T}M_0}{2}\Big)^i\\
&= \frac{\frac{e^{2\lambda T}}{2}\int\limits_0^T p(t)\,dt}{1-\frac{e^{2\lambda T}M_0}{2}}\,.
\end{aligned} \tag{4.6}$$

On the other hand, from (4.2), (4.5) and (4.6), for almost all $t \in I$, we obtain

$$\begin{aligned}\big|f_n(t) - D^{\rho}_{CF}y(t) - \lambda y(t)\big| &\le \sum_{i=1}^{n-1} |f_{i+1}(t) - f_i(t)| + \big|f_1(t) - D^{\rho}_{CF}y(t) - \lambda y(t)\big| \\ &\le L(t)\,\frac{e^{2\lambda T}\int\limits_0^T p(t)\,dt}{2 - M_0 e^{2\lambda T}} + p(t).\end{aligned}$$

Hence the sequence $f_n(\cdot)$ is integrable bounded and therefore $f(\cdot) \in L^1(I,\mathbf{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (4.3), (4.4), we deduce that $x(\cdot)$ is a solution of (1.1). Finally, passing to the limit in (4.6), we obtain the required estimate on $x(\cdot)$.

It remains to construct the sequences $x_n(\cdot)$, $f_n(\cdot)$ with the properties (4.3)–(4.5). The construction will be done by induction.

Since the first step is already realized, assume that for some $N \ge 1$, we already constructed $x_n(\cdot) \in C(I,\mathbf{R})$ and $f_n(\cdot) \in L^1(I,\mathbf{R})$, $n = 1,2,\dots,N$, satisfying (4.3), (4.5) for $n = 1,2,\dots N$ and (4.4) for $n = 1,2,\dots,N-1$. The set-valued map $t \to F(t, x_N(t), V(x_N)(t))$ is measurable. Moreover, the map

$$t \to L(t)\bigg(|x_N(t) - x_{N-1}(t)| + \int\limits_0^t L(s)|x_N(s) - x_{N-1}(s)|\,ds\bigg)$$

is measurable. By the lipschitzianity of $F(t,\cdot)$, for almost all $t \in I$, we have

$$F(t, x_N(t)) \cap \bigg\{f_N(t) + L(t)\bigg(|x_N(t) - x_{N-1}(t)| + \int\limits_0^t L(s)|x_N(s) - x_{N-1}(s)|\,ds\bigg)[-1,1]\bigg\} \neq \varnothing.$$

Lemma 2.1 yields that there exists a measurable selection $f_{N+1}(\cdot)$ of $F(\cdot, x_N(\cdot), V(x_N)(\cdot))$ such that

$$|f_{N+1}(t) - f_N(t)| \le L(t)\bigg(|x_N(t) - x_{N-1}(t)| + \int\limits_0^t L(s)|x_N(s) - x_{N-1}(s)|\,ds\bigg)$$

for almost all $t \in I$.

We define $x_{N+1}(\cdot)$ as in (4.3) with $n = N+1$. Thus $f_{N+1}(\cdot)$ satisfies (4.4) and (4.5) and the proof is complete. □

The assumptions in Theorem 4.1 are satisfied, in particular, for $y(\cdot) = 0$ and therefore with $p(\cdot) = L(\cdot)$. We obtain the following consequence of Theorem 4.1.

Corollary 4.1. *Assume that Hypothesis* H3 *is satisfied,* $e^{2\lambda T}M_0 < 2$ *and* $d(0, F(t,0,V(0)(t)) \le L(t)$ *a.e.* (I). *Then there exists a solution* $x(\cdot)$ *of problem* (4.1) *satisfying*

$$|x(t)| \le \frac{e^{2\lambda T}}{2 - M_0 e^{2\lambda T}} \int\limits_0^T L(t)\,dt \ \text{ for all } \ t \in I.$$

If F does not depend on the last variable, Hypothesis H3 becomes

Hypothesis H4.

(i) $F(\cdot,\cdot) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ measurable.

(ii) There exists $L(\cdot) \in L^1(I,(0,\infty))$ such that for almost all $t \in I$, $F(t,\cdot)$ is $L(t)$-Lipschitz in the sense that

$$d_H\big(F(t,x_1), F(t,x_2)\big) \le L(t)|x_1 - x_2|, \quad \forall\, x_1, x_2 \in \mathbf{R}.$$

Denote

$$L_0 = \int_0^T L(t)\,dt.$$

Corollary 4.2. *Assume that Hypothesis* H4 *is satisfied,* $e^{2\lambda T} L_0 < 2$ *and* $d(0, F(t,0)) \le L(t)$ *a.e.* (I). *Then there exists a solution* $x(\cdot)$ *of problem* (1.1) *satisfying*

$$|x(t)| \le \frac{e^{2\lambda T} L_0}{2 - e^{2\lambda T} L_0} \quad \textit{for all } t \in I. \tag{4.7}$$

Remark. A result similar to Corollary 4.2 can be obtained by using the set-valued contraction principle in [13]. Namely, we define $M : C(I, \mathbf{R}) \to \mathcal{P}(C(I, \mathbf{R}))$ by

$$M(x) := \left\{ v(\cdot) \in C(I, \mathbf{R}) :\ v(t) = \int_0^T G(t,s) f(s)\,ds,\ f \in S_F(x) \right\}.$$

It can be shown that $M(x) \ne \varnothing$ for any $x \in C(I, \mathbf{R})$, $M(x)$ is closed for any $x \in C(I, \mathbf{R})$ and $M(\cdot)$ is $\frac{e^{2\lambda T} L_0}{2}$-contraction. Therefore, by the Covitz–Nadler set-valued contraction principle, $M(\cdot)$ has a fixed point which is a solution to problem (1.1).

Unfortunately, this approach does not contain a priori bounds for solutions as in (4.7).

Example.

(a) Consider $\rho = \frac{1}{2}$, $\lambda = \frac{1}{2}$, $T = 1$, define $F(\cdot,\cdot) : [0,1] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ by

$$F(t,x) = \Big[0, \frac{|x|}{1+|x|}\Big].$$

Obviously, Hypothesis H1 is satisfied with $\varphi = 1$, $\psi(z) \equiv \frac{1}{1+z}$, and consider $r > \frac{e}{2} - 1$. Therefore, by Theorem 3.1, there exists a solution of problem

$$D_{CF}^{\frac{1}{2}} x(t) + \frac{1}{2}\, x(t) \in \Big[0, \frac{|x(t)|}{1+|x(t)|}\Big], \quad x(0) = -x(T)$$

such that $|x(t)| \le r$, $\forall\, t \in [0,1]$.

(b) Consider $\rho = \frac{1}{2}$, $\lambda = \frac{1}{2}$, $T = 1$, define $F(\cdot,\cdot) : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ by

$$F(t,x,y) = \Big[-\frac{1}{10} \cdot \frac{|x|}{1+|x|}, 0\Big] \cup \Big[0, \frac{1}{10} \cdot \frac{|y|}{1+|y|}\Big]$$

and $k(\cdot,\cdot,\cdot) : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ by $k(t,s,x) = \frac{1}{10}\, x$.

Since

$$\sup\big\{|u| :\ u \in F(t,x,y)\big\} \le \frac{1}{10}, \quad \forall\, t \in [0,1], \quad x, y \in \mathbf{R},$$

$$\mathrm{d}_H\big(F(t,x_1,y_1), F(t,x_2,y_2)\big) \le \frac{1}{10}\, |x_1 - x_2| + \frac{1}{10}\, |y_1 - y_2|, \quad \forall\, x_1, x_2, y_1, y_2 \in \mathbf{R},$$

in this case

$$p(t) \equiv L(t) \equiv \frac{1}{10}, \quad M(t) = \frac{1}{10}\Big(1 + \frac{1}{10}\, t\Big) \text{ and } M_0 = \frac{1}{10} + \frac{1}{2}\Big(\frac{1}{10}\Big)^2.$$

Since

$$e\Big(\frac{1}{10} + \frac{1}{2}\Big(\frac{1}{10}\Big)^2\Big) < 2,$$

we apply Corollary 4.1 in order to deduce the existence of a solution of the problem

$$D^{\frac{1}{2}}_{CF}x(t)+\frac{1}{2}\,x(t)\in\Big[-\frac{1}{10}\cdot\frac{|x(t)|}{1+|x(t)|},0\Big]\cup\Bigg[0,\frac{1}{10}\cdot\frac{|\int\limits_0^t x(s)\,ds|}{10+|\int\limits_0^t x(s)\,ds|}\Bigg],\quad x(0)=-x(T)$$

that satisfies

$$|x(t)|\leq\frac{20e}{400-21e}\,,\ \ \forall\,t\in[0,1].$$

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Author's address:

Aurelian Cernea

1. Faculty of Mathematics and Computer Science, University of Bucharest, Academiei 14, Bucharest 010014, Romania.

2. Academy of Romanian Scientists, Ilfov 3, Bucharest 050044, Romania.

E-mail: `acernea@fmi.unibuc.ro`