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A STUDY OF A SYSTEM OF NONLINEAR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH NEW INTEGRAL BOUNDARY CONDITIONS

Abstract. We investigate the existence of solutions for a coupled system of nonlinear Caputo fractional differential equations equipped with a new class of integral boundary conditions. The existence of solutions for the given problem is shown with the aid of the Leray–Schauder nonlinear alternative, while the Banach fixed point theorem is applied to establish a uniqueness result for the given problem. An example is formulated to demonstrate the application of main results.

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1 Introduction

In this paper, we consider a new boundary value problem consisting of a system of nonlinear Caputo fractional differential equations and closed type integral boundary conditions. Precisely, we discuss the existence of solutions for the following problem:

$$\begin{cases} {}^C D^{q_1}\varphi(t) = \rho_1(t,\varphi(t),\psi(t)), \quad 2 < q_1 < 3, \;\; t \in \mathcal{J} = [0,T], \;\; T > 0, \\ {}^C D^{q_2}\psi(t) = \rho_2(t,\varphi(t),\psi(t)), \quad 2 < q_2 < 3, \;\; t \in \mathcal{J} = [0,T], \;\; T > 0, \end{cases} \tag{1.1}$$

$$\begin{cases} \varphi(0) = 0, \quad \varphi'(T) = 0, \quad \varphi(T) = \int_0^{\xi} \big[p_1\psi(s) + Tp_2\psi'(s)\big]\, ds, \quad p_1, p_2 \in \mathbb{R}, \\ \psi(0) = 0, \quad \psi'(T) = 0, \quad \psi(T) = \int_0^{\xi} \big[r_1\varphi(s) + Tr_2\varphi'(s)\big]\, ds, \quad r_1, r_2 \in \mathbb{R}, \end{cases} \tag{1.2}$$

where ${}^C D^{q_i}$ denotes the Caputo fractional derivatives of order q_i $(i = 1,2)$ (here, ${}^C D^{q_i}$ is used instead of ${}^C D^{q_i}_{0+}$ for the sake of convenience), $\rho_1, \rho_2 \in C(\mathcal{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\xi \in (0,T)$.

Here, we emphasize that the first integral boundary condition in (1.2) can be interpreted as the value of the unknown function φ at the right end T of the interval $[0,T]$, is proportional to the linear combination of the unknown function ψ and its derivative ψ' on the sub-interval $[0,\xi]$ of $[0,T]$. The second integral boundary condition can be interpreted in a similar manner by interchanging the role of φ and ψ. It is worthwhile to mention that the involvement of both unknown functions and their derivatives in the integral boundary conditions makes the proposed problem novel and useful. It can be noticed that the integral boundary conditions given in (1.2) can be written as

$$\varphi(T) = p_1 \int_0^{\xi-} \psi(s)\, ds + Tp_2\psi(\xi), \quad \psi(T) = r_1 \int_0^{\xi-} \varphi(s)\, ds + Tr_2\varphi(\xi).$$

Now, we illustrate the application of integral boundary conditions and systems of fractional differential equations. Concerning the importance of integral boundary conditions, it is imperative to point out that such boundary conditions play an important role in the investigation of problems of applied nature. In case of fluid flow problems [32, 36, 45], the idea of integral boundary conditions serves as an effective tool to describe the boundary data on arbitrary shaped structures as the assumption of circular cross-section cannot be justified on such (curved) structures. Furthermore, integral boundary conditions appear in the study of thermal conduction [8], semiconductor [22], hydrodynamic problems [10], diffusion problems [7], bacterial self-regularization [11], etc. For some interesting results on boundary value problems with integral boundary conditions, see, e.g., [3, 20, 38] and the references cited therein.

Let us now review some recent literature on fractional differential equations. In recent years, there has been observed a great surge in developing the topic of fractional calculus because of its extensive application in the mathematical modeling of several scientific and technical phenomena. Examples include immune system with memory [13], co-infection of malaria and HIV/AIDS [9], chaos and fractional dynamics [43], stabilization of chaotic systems [15], chaotic synchronization [41, 44], dynamical networks with multiple weights [42], neural networks [34], phytoplankton-zooplankton systems [24], economic model [35], financial economics [16], etc.

As the mathematical models associated with physical phenomena contain initial and boundary value problems, many researchers have shown a keen interest in developing the theoretical aspects of such problems (see, e.g., the books [4, 5] and a series of research articles [1, 23, 26, 28, 29, 33, 39]). Keeping in mind the occurrence of fractional differential systems in the mathematical modeling of several real world problems [12, 19, 37, 40], many investigators studied such systems with a variety of boundary conditions (see, e.g., [14, 18, 21, 25, 30, 31]). For some works on fractional differential systems with closed boundary conditions, see, e.g., [2, 6] and the references cited therein.

Our study in the present research paper is motivated by the application of integral boundary conditions and systems of fractional differential equations in a variety of physical and technical situations.

We prove the existence and uniqueness of solutions to the problem (1.1), (1.2) with the aid of standard fixed point theorems. The work established in this paper is new and contributes to the literature on fractional-order systems with integral boundary conditions.

The remainder of the paper is arranged as follows. We collect some preliminary definitions of fractional calculus and solve a linear version of problem (1.1), (1.2) in Section 2 The main results and examples illustrating these results are presented in Section 3. The concluding remarks and some special cases arising from the present study are elaborated in Section 4.

2 A preliminary result

Before proceeding for a preliminary result dealing with the linear version of problem (1.1), (1.2), we enlist the related definitions from fractional calculus.

Definition 2.1 ([27])**.** For a locally integrable real-valued function σ defined on $-\infty \le a < t < b \le +\infty$, the (left) Riemann–Liouville fractional integral of order $\omega \in \mathbb{R}^+$, denoted by $I^{\omega}_{a^+}\sigma$, is given by

$$I^{\omega}_{a^+}\sigma(t) = \frac{1}{\Gamma(\omega)} \int_a^t (t-s)^{\omega-1}\sigma(s)\,ds,$$

where Γ denotes the Euler gamma function.

Remark 2.1. For $\alpha, \beta \in \mathbb{R}^+$, $x \in [a,b]$ and $\sigma \in L_p(a,b)$ $(1 \le p \le \infty)$, the (left) Riemann–Liouville fractional integral operators satisfy the following relation:

$$(I^{\alpha}_{a+} I^{\beta}_{a+}\sigma)(x) = (I^{\alpha+\beta}_{a+}\sigma)(x)$$

for almost every point $x \in [a,b]$. If $\alpha + \beta > 1$, then the above relation holds at any point of $[a,b]$.

Definition 2.2 ([27])**.** The (left) Riemann–Liouville fractional derivative $D^{\omega}_{a^+}\sigma$ of order $\omega \in (m-1, m]$, $m \in \mathbb{N}$, is defined as

$$D^{\omega}_{a^+}\sigma(t) = \frac{1}{\Gamma(m-\omega)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\omega}\sigma(s)\,ds,$$

where $\sigma, \sigma^{(m)} \in L^1[a,b]$ for $-\infty \le a < t < b \le +\infty$.

Definition 2.3 ([27])**.** The (left) Caputo fractional derivative ${}^C D^{\omega}_{a^+}\sigma$ of order $\omega \in (m-1, m]$, $m \in \mathbb{N}$, in terms of Riemann–Liouville fractional derivative operator $D^{\omega}_{a^+}$, is defined as

$${}^C D^{\omega}_{a^+}\sigma(t) = D^{\omega}_{a^+}\Big[\sigma(t) - \sigma(a) - \sigma'(a)\,\frac{(t-a)}{1!} - \cdots - \sigma^{(m-1)}(a)\,\frac{(t-a)^{m-1}}{(m-1)!}\Big].$$

Remark 2.2. The (left) Caputo fractional derivative of order $\omega \in (m-1, m]$, $m \in \mathbb{N}$, for a continuous function $\sigma : [a,b] \to \mathbb{R}$ such that $\sigma \in C^m[a,b]$, existing almost everywhere on $[a,b]$, is defined by

$${}^C D^{\omega}_{a^+}\sigma(t) = \frac{1}{\Gamma(m-\omega)} \int_a^t (t-s)^{m-\omega-1}\sigma^{(m)}(s)\,ds.$$

In the following lemma, we solve the linear variant of problem (1.1), (1.2). This lemma plays a key role in converting the given problem into a fixed point problem.

Lemma 2.1. *For $F, G \in C(\mathcal{J}, \mathbb{R})$ and $\Delta \neq 0$, the unique solution of the linear system*

$$\begin{cases} {}^C D^{q_1}\varphi(t) = F(t), & t \in \mathcal{J}, \\ {}^C D^{q_2}\psi(t) = G(t), & t \in \mathcal{J}, \\ \varphi(0) = 0, \quad \varphi'(T) = 0, \quad \varphi(T) = \int_0^\xi [p_1\psi(s) + Tp_2\psi'(s)]\,ds, & p_1, p_2 \in \mathbb{R}, \\ \psi(0) = 0, \quad \psi'(T) = 0, \quad \psi(T) = \int_0^\xi [r_1\varphi(s) + Tr_2\varphi'(s)]\,ds, & r_1, r_2 \in \mathbb{R}, \end{cases} \tag{2.1}$$

is given by the following pair of integral equations:

$$\begin{aligned} \varphi(t) = \int_0^t \frac{(t-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)\,d\upsilon - \frac{1}{\Delta}\Bigg\{ \alpha_1(t) \int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)} F(\upsilon)\,d\upsilon \\ + \alpha_2(t)\Bigg(p_1 \int_0^\xi \frac{(\xi-\upsilon)^{q_2}}{\Gamma(q_2+1)} G(\upsilon)\,d\upsilon + Tp_2 \int_0^\xi \frac{(\xi-\upsilon)^{q_2-1}}{\Gamma(q_2)} G(\upsilon)\,d\upsilon\,ds - \int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)\,d\upsilon \Bigg) \\ + \alpha_3(t) \int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)} G(\upsilon)\,d\upsilon \\ + \alpha_4(t)\Bigg(r_1 \int_0^\xi \frac{(\xi-\upsilon)^{q_1}}{\Gamma(q_1+1)} F(\upsilon)\,d\upsilon + Tr_2 \int_0^\xi \frac{(\xi-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)\,d\upsilon - \int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)} G(\upsilon)\,d\upsilon \Bigg) \Bigg\}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \psi(t) = \int_0^t \frac{(t-\upsilon)^{q_2-1}}{\Gamma(q_2)} G(\upsilon)\,d\upsilon - \frac{1}{\Delta}\Bigg\{ \beta_1(t) \int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)} F(\upsilon)\,d\upsilon \\ + \beta_2(t)\Bigg(p_1 \int_0^\xi \frac{(\xi-\upsilon)^{q_2}}{\Gamma(q_2+1)} G(\upsilon)\,d\upsilon + Tp_2 \int_0^\xi \frac{(\xi-\upsilon)^{q_2-1}}{\Gamma(q_2)} G(\upsilon)\,d\upsilon - \int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)\,d\upsilon \Bigg) \\ + \beta_3(t) \int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)} G(\upsilon)\,d\upsilon \\ + \beta_4(t)\Bigg(r_1 \int_0^\xi \frac{(\xi-\upsilon)^{q_1}}{\Gamma(q_1+1)} F(\upsilon)\,d\upsilon + Tr_2 \int_0^\xi \frac{(\xi-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)\,d\upsilon - \int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)} G(\upsilon)\,d\upsilon \Bigg) \Bigg\}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \alpha_1(t) &= \big[2TA_1B_2 - T^4 - A_2B_2\big]t + \big[T^3 - 2TA_1B_1 + A_2B_1\big]t^2, \\ \alpha_2(t) &= -2T^3t + T^2t^2, \\ \alpha_3(t) &= \big[-2T^3A_1 + 2T^2A_2\big]t + \big[T^2A_1 - TA_2\big]t^2, \\ \alpha_4(t) &= \big[-4T^2A_1 + 2TA_2\big]t + \big[2TA_1 - A_2\big]t^2, \\ \beta_1(t) &= \big[2T^2B_2 - 2T^3B_1\big]t + \big[T^2B_1 - TB_2\big]t^2, \\ \beta_2(t) &= \big[2TB_2 - 4T^2B_1\big]t + \big[2TB_1 - B_2\big]t^2, \\ \beta_3(t) &= \big[-T^4 - A_2B_2 + 2TA_2B_1\big]t + \big[T^3 + A_1B_2 - 2TA_1B_1\big]t^2, \\ \beta_4(t) &= -2T^3t + T^2t^2, \end{aligned} \tag{2.4}$$

and

$$\Delta = T^4 + 2TA_1B_2 - A_2B_2 - 4T^2A_1B_1 + 2TA_2B_1,$$

with

$$A_1 = p_1\frac{\xi^2}{2} + Tp_2\xi, \quad A_2 = p_1\frac{\xi^3}{3} + Tp_2\xi^2, \quad B_1 = r_1\frac{\xi^2}{2} + Tr_2\xi, \quad B_2 = r_1\frac{\xi^3}{3} + Tr_2\xi^2. \tag{2.5}$$

Proof. For arbitrary constants $c_0, c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$, it is well-known (see [27]) that the solution of fractional differential equations in (2.1) can be written as

$$\varphi(t) = \int_0^t \frac{(t-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)\,d\upsilon - c_0 - c_1 t - c_2 t^2, \tag{2.6}$$

$$\psi(t) = \int_0^t \frac{(t-\upsilon)^{q_2-1}}{\Gamma(q_2)} G(\upsilon)\,d\upsilon - c_3 - c_4 t - c_5 t^2. \tag{2.7}$$

Using (2.6), (2.7) in the boundary conditions of problem (2.1) yields $c_0 = 0$, $c_3 = 0$ and a system of algebraic equations in c_1, c_2, c_4 and c_5 given by

$$\begin{aligned} c_1 + 2Tc_2 &= \int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)} F(\upsilon)\,d\upsilon, \quad -Tc_1 - T^2c_2 + A_1c_4 + A_2c_5 = A_3, \\ c_4 + 2Tc_5 &= \int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)} G(\upsilon)\,d\upsilon, \quad B_1c_1 + B_2c_2 - Tc_4 - T^2c_5 = B_3, \end{aligned} \tag{2.8}$$

where A_1, A_2, B_1, B_2 are given in (2.5) and

$$\begin{aligned} A_3 &= p_1\int_0^\xi \frac{(\xi-\upsilon)^{q_2}}{\Gamma(q_2+1)} G(\upsilon)\,d\upsilon + Tp_2\int_0^\xi \frac{(\xi-\upsilon)^{q_2-1}}{\Gamma(q_2)} G(\upsilon)\,d\upsilon - \int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)\,d\upsilon, \\ B_3 &= r_1\int_0^\xi \frac{(\xi-\upsilon)^{q_1}}{\Gamma(q_1+1)} F(\upsilon)\,d\upsilon + Tr_2\int_0^\xi \frac{(\xi-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)\,d\upsilon\,ds - \int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)} G(\upsilon)\,d\upsilon. \end{aligned}$$

Solving system (2.8) for c_1, c_2, c_4 and c_5, we obtain

$$\begin{aligned} c_1 = \frac{1}{\Delta}\Bigg\{ &\big[2TA_1B_2 - T^4 - A_2B_2\big]\int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)} F(\upsilon)\,d\upsilon \\ &- 2T^3\Bigg(p_1\int_0^\xi \frac{(\xi-\upsilon)^{q_2}}{\Gamma(q_2+1)} G(\upsilon)\,d\upsilon + Tp_2\int_0^\xi \frac{(\xi-\upsilon)^{q_2-1}}{\Gamma(q_2)} G(\upsilon)\,d\upsilon - \int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)\,d\upsilon\Bigg) \\ &+ \big[2T^2A_2 - 2T^3A_1\big]\int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)} G(\upsilon)\,d\upsilon + \big[2TA_2 - 4T^2A_1\big]\Bigg(r_1\int_0^\xi \frac{(\xi-\upsilon)^{q_1}}{\Gamma(q_1+1)} F(\upsilon)\,d\upsilon \\ &+ Tr_2\int_0^\xi \frac{(\xi-\upsilon)^{q_1-1}}{\Gamma(q_1)} F(\upsilon)\,d\upsilon\,ds - \int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)} G(\upsilon)\,d\upsilon\Bigg)\Bigg\}, \end{aligned}$$

$$c_2 = \frac{1}{\Delta}\Bigg\{\big[T^3 - 2TA_1B_1 + A_2B_1\big]\int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)} F(\upsilon)\,d\upsilon$$

$$+T^2\Big(p_1\int_0^{\xi}\frac{(\xi-\upsilon)^{q_2}}{\Gamma(q_2+1)}G(\upsilon)\,d\upsilon+Tp_2\int_0^{\xi}\frac{(\xi-\upsilon)^{q_2-1}}{\Gamma(q_2)}G(\upsilon)\,d\upsilon-\int_0^{T}\frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}F(\upsilon)\,d\upsilon\Big)$$

$$+\big[T^2A_1-TA_2\big]\int_0^{T}\frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}G(\upsilon)\,d\upsilon+\big[2TA_1-A_2\big]\Big(r_1\int_0^{\xi}\frac{(\xi-\upsilon)^{q_1}}{\Gamma(q_1+1)}F(\upsilon)\,d\upsilon$$

$$+Tr_2\int_0^{\xi}\frac{(\xi-\upsilon)^{q_1-1}}{\Gamma(q_1)}F(\upsilon)\,d\upsilon\,ds-\int_0^{T}\frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}G(\upsilon)\,d\upsilon\Big)\Big\},$$

$$c_4=\frac{1}{\Delta}\Big\{\big[2T^2B_2-2T^3B_1\big]\int_0^{T}\frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}F(\upsilon)\,d\upsilon$$

$$+\big[2TB_2-4T^2B_1\big]\Big(p_1\int_0^{\xi}\frac{(\xi-\upsilon)^{q_2}}{\Gamma(q_2+1)}G(\upsilon)\,d\upsilon+Tp_2\int_0^{\xi}\frac{(\xi-\upsilon)^{q_2-1}}{\Gamma(q_2)}G(\upsilon)\,d\upsilon$$

$$-\int_0^{T}\frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}F(\upsilon)\,d\upsilon\Big)+\big[2TA_2B_1-T^4-A_2B_2\big]\int_0^{T}\frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}G(\upsilon)\,d\upsilon$$

$$-2T^3\Big(r_1\int_0^{\xi}\frac{(\xi-\upsilon)^{q_1}}{\Gamma(q_1+1)}F(\upsilon)\,d\upsilon+Tr_2\int_0^{\xi}\frac{(\xi-\upsilon)^{q_1-1}}{\Gamma(q_1)}F(\upsilon)\,d\upsilon\,ds-\int_0^{T}\frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}G(\upsilon)\,d\upsilon\Big)\Big\},$$

$$c_5=\frac{1}{\Delta}\Big\{\big[T^2B_1-TB_2\big]\int_0^{T}\frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}F(\upsilon)\,d\upsilon$$

$$+\big[2TB_1-B_2\big]\Big(p_1\int_0^{\xi}\frac{(\xi-\upsilon)^{q_2}}{\Gamma(q_2+1)}G(\upsilon)\,d\upsilon+Tp_2\int_0^{\xi}\frac{(\xi-\upsilon)^{q_2-1}}{\Gamma(q_2)}G(\upsilon)\,d\upsilon$$

$$-\int_0^{T}\frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}F(\upsilon)\,d\upsilon\Big)+\big[T^3+A_1B_2-2TA_1B_1\big]\int_0^{T}\frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}G(\upsilon)\,d\upsilon$$

$$+T^2\Big(r_1\int_0^{\xi}\frac{(\xi-\upsilon)^{q_1}}{\Gamma(q_1+1)}F(\upsilon)\,d\upsilon+Tr_2\int_0^{\xi}\frac{(\xi-\upsilon)^{q_1-1}}{\Gamma(q_1)}F(\upsilon)\,d\upsilon\,ds-\int_0^{T}\frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}G(\upsilon)\,d\upsilon\Big)\Big\},$$

Inserting the above values of c_i, $i=1,2,4,5$, in (2.6), (2.7) together with (2.4), we get solution (2.2), (2.3). By direct computation, one can obtain the converse of the lemma. □

3 Main results

Let Θ denote the Banach space of all continuous functions from $\mathcal{J}$ to $\mathbb{R}$ equipped with the supremum norm: $\|\vartheta\|=\sup_{t\in\mathcal{J}}|\vartheta(t)|$, $\vartheta\in\Theta$. Then the product space $\Theta\times\Theta$ is also a Banach space endowed with the norm

$$\|(\vartheta_1,\vartheta_2)\|=\|\vartheta_1\|+\|\vartheta_2\|,\ \ (\vartheta_1,\vartheta_2)\in\Theta\times\Theta.$$

Associated with the nonlinear problem (1.1), (1.2), we define an operator $\mathcal{H}:\Theta\times\Theta\to\Theta\times\Theta$ by

$$\mathcal{H}(\varphi,\psi)(t)=\begin{pmatrix}\mathcal{H}_1(\varphi,\psi)(t)\\ \mathcal{H}_2(\varphi,\psi)(t)\end{pmatrix}, \tag{3.1}$$

where

$$\begin{aligned}\mathcal{H}_1(\varphi,\psi)(t) = &\int\limits_0^t \frac{(t-\upsilon)^{q_1-1}}{\Gamma(q_1)}\,\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon - \frac{1}{\Delta}\Bigg\{\alpha_1(t)\int\limits_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}\,\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\\
&+\alpha_2(t)\Bigg(p_1\int\limits_0^\xi \frac{(\xi-\upsilon)^{q_2}}{\Gamma(q_2+1)}\,\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon + Tp_2\int\limits_0^\xi \frac{(\xi-\upsilon)^{q_2-1}}{\Gamma(q_2)}\,\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\,ds\\
&-\int\limits_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}\,\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\Bigg) + \alpha_3(t)\int\limits_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}\,\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\\
&+\alpha_4(t)\Bigg(r_1\int\limits_0^\xi \frac{(\xi-\upsilon)^{q_1}}{\Gamma(q_1+1)}\,\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon + Tr_2\int\limits_0^\xi \frac{(\xi-\upsilon)^{q_1-1}}{\Gamma(q_1)}\,\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\\
&-\int\limits_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}\,\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\Bigg)\Bigg\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}_2(\varphi,\psi)(t) = &\int\limits_0^t \frac{(t-\upsilon)^{q_2-1}}{\Gamma(q_2)}\,\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon - \frac{1}{\Delta}\Bigg\{\beta_1(t)\int\limits_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}\,\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\\
&+\beta_2(t)\Bigg(p_1\int\limits_0^\xi \frac{(\xi-\upsilon)^{q_2}}{\Gamma(q_2+1)}\,\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon + Tp_2\int\limits_0^\xi \frac{(\xi-\upsilon)^{q_2-1}}{\Gamma(q_2)}\,\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\\
&-\int\limits_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}\,\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\Bigg) + \beta_3(t)\int\limits_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}\,\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\\
&+\beta_4(t)\Bigg(r_1\int\limits_0^\xi \frac{(\xi-\upsilon)^{q_1}}{\Gamma(q_1+1)}\,\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon + Tr_2\int\limits_0^\xi \frac{(\xi-\upsilon)^{q_1-1}}{\Gamma(q_1)}\,\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\\
&-\int\limits_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}\,\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\Bigg)\Bigg\}.\end{aligned}$$

Here, it is imperative to mention that problem (1.1), (1.2) is equivalent to the fixed point problem: $\mathcal{H}(\varphi,\psi) = (\varphi,\psi)$, where the operator $\mathcal{H} : \Theta\times\Theta \to \Theta\times\Theta$ is defined by (3.1). This means that a fixed point of the operator $\mathcal{H}$ will be a solution to problem (1.1), (1.2).

To analyze problem (1.1), (1.2), we need the following hypotheses:

(H_1) For $\rho_1, \rho_2 \in C(\mathcal{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, there exist real constants $m_i,\ n_i \geq 0,\ i = 1,2$, and $m_0, n_0 > 0$ such that

$$|\rho_1(t,\varphi,\psi)| \leq m_0 + m_1|\varphi| + m_2|\psi|, \quad |\rho_2(t,\varphi,\psi)| \leq n_0 + n_1|\varphi| + n_2|\psi|, \ \ \forall\, \varphi,\psi\in\mathbb{R}.$$

(H_2) There exist the constants $\ell_i, \overline{\ell}_i,\ i = 1,2$, such that for all $t\in\mathcal{J},\ \varphi_i,\psi_i\in\mathbb{R},\ i = 1,2$,

$$\begin{aligned}\big|\rho_1(t,\varphi_1,\psi_1) - \rho_1(t,\varphi_2,\psi_2)\big| &\leq \ell_1|\varphi_1-\varphi_2| + \ell_2|\psi_1-\psi_2|,\\
\big|\rho_2(t,\varphi_1,\psi_1) - \rho_2(t,\varphi_2,\psi_2)\big| &\leq \overline{\ell}_1|\varphi_1-\varphi_2| + \overline{\ell}_2|\psi_1-\psi_2|.\end{aligned}$$

For computational convenience, we set the notation:

$$
\begin{aligned}
\mathcal{Q}_1 &= \max_{t\in[0,T]}\left\{\frac{t^{q_1}}{\Gamma(q_1+1)}+\frac{1}{|\Delta|}\left[|\alpha_1(t)|\,\frac{T^{q_1-1}}{\Gamma(q_1)}+|\alpha_2(t)|\,\frac{T^{q_1}}{\Gamma(q_1+1)}\right.\right.\\
&\qquad\left.\left.+|\alpha_4(t)|\Big(|r_1|\,\frac{\xi^{q_1+1}}{\Gamma(q_1+2)}+|r_2|\,\frac{T\xi^{q_1}}{\Gamma(q_1+1)}\Big)\right]\right\},\\
\mathcal{Q}_2 &= \max_{t\in[0,T]}\frac{1}{|\Delta|}\left\{|\alpha_2(t)|\Big(|p_1|\,\frac{\xi^{q_2+1}}{\Gamma(q_2+2)}+|p_2|\,\frac{T\xi^{q_2}}{\Gamma(q_2+1)}\Big)\right.\\
&\qquad\left.+|\alpha_3(t)|\,\frac{T^{q_2-1}}{\Gamma(q_2)}+|\alpha_4(t)|\,\frac{T^{q_2}}{\Gamma(q_2+1)}\right\},\\
\mathcal{Q}_3 &= \max_{t\in[0,T]}\left\{\frac{t^{q_2}}{\Gamma(q_2+1)}+\frac{1}{|\Delta|}\left[|\beta_2(t)|\Big(|p_1|\,\frac{\xi^{q_2+1}}{\Gamma(q_2+2)}+|p_2|\,\frac{T\xi^{q_2}}{\Gamma(q_2+1)}\Big)\right.\right.\\
&\qquad\left.\left.+|\beta_3(t)|\,\frac{T^{q_2-1}}{\Gamma(q_2)}+|\beta_4(t)|\,\frac{T^{q_2}}{\Gamma(q_2+1)}\right]\right\},\\
\mathcal{Q}_4 &= \max_{t\in[0,T]}\frac{1}{|\Delta|}\left\{|\beta_1(t)|\,\frac{T^{q_1-1}}{\Gamma(q_1)}+|\beta_2(t)|\,\frac{T^{q_1}}{\Gamma(q_1+1)}\right.\\
&\qquad\left.+|\beta_4(t)|\Big(|r_1|\,\frac{\xi^{q_1+1}}{\Gamma(q_1+2)}+|r_2|\,\frac{T\xi^{q_1}}{\Gamma(q_1+1)}\Big)\right\}
\end{aligned}
\tag{3.2}
$$

and

$$
\mathcal{Q}_0=\min\Big\{1-\big[m_1(\mathcal{Q}_1+\mathcal{Q}_4)+n_1(\mathcal{Q}_2+\mathcal{Q}_3)\big],\,1-\big[m_2(\mathcal{Q}_1+\mathcal{Q}_4)+n_2(\mathcal{Q}_2+\mathcal{Q}_3)\big]\Big\}. \tag{3.3}
$$

Now, we proceed to present our main results. In our first result, we prove the existence of solutions for problem (1.1), (1.2) by using the Leray–Schauder nonlinear alternative [17].

Theorem 3.1. *Assume that the condition (H_1) is satisfied. Then there exists at least one solution to problem* (1.1), (1.2) *on $\mathcal{J}$, provided that*

$$
m_1(\mathcal{Q}_1+\mathcal{Q}_4)+n_1(\mathcal{Q}_2+\mathcal{Q}_3)<1,\quad m_2(\mathcal{Q}_1+\mathcal{Q}_4)+n_2(\mathcal{Q}_2+\mathcal{Q}_3)<1,
$$

where $\mathcal{Q}_i$, $i=1,2,3,4$ are given in (3.2).

Proof. We verify the hypotheses of the Leray-Schauder nonlinear alternative [17] in different steps. Let us first show that the operator $\mathcal{H}:\Theta\times\Theta\to\Theta\times\Theta$ defined by (3.1) is completely continuous. Note that the operator $\mathcal{H}$ is continuous in view of continuity of the functions ρ_1 and ρ_2. If $\Upsilon\subset\Theta\times\Theta$ is bounded, then there exist the positive constants L_1 and L_2 such that $|\rho_1(t,\varphi,\psi)|\le L_1$, $|\rho_2(t,\varphi,\psi)|\le L_2$, $\forall\,(\varphi,\psi)\in\Upsilon$. Then, for any $(\varphi,\psi)\in\Upsilon$, we obtain

$$
\begin{aligned}
\|\mathcal{H}_1(\varphi,\psi)\| &\le \max_{t\in\mathcal{J}}\left\{\int_0^t\frac{(t-\upsilon)^{q_1-1}}{\Gamma(q_1)}\big|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon\right.\\
&+\frac{1}{|\Delta|}\left[|\alpha_1(t)|\int_0^T\frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}\big|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon+|\alpha_2(t)|\left(|p_1|\int_0^\xi\frac{(\xi-\upsilon)^{q_2}}{\Gamma(q_2+1)}\big|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon\right.\right.\\
&\left.+T|p_2|\int_0^\xi\frac{(\xi-\upsilon)^{q_2-1}}{\Gamma(q_2)}\big|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon+\int_0^T\frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}\big|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon\right)\\
&+|\alpha_3(t)|\int_0^T\frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}\big|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon+|\alpha_4(t)|\left(|r_1|\int_o^\xi\frac{(\xi-\upsilon)^{q_1}}{\Gamma(q_1+1)}\big|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon\right.
\end{aligned}
$$

$$
\begin{aligned}
&+T|r_2|\int_0^{\xi}\frac{(\xi-\upsilon)^{q_1-1}}{\Gamma(q_1)}\big|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon+\int_0^{T}\frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}\big|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon\Big)\Big]\Big\}\\
&\le L_1\max_{t\in[0,T]}\Big\{\frac{t^{q_1}}{\Gamma(q_1+1)}+\frac{1}{|\Delta|}\Big[|\alpha_1(t)|\frac{T^{q_1-1}}{\Gamma(q_1)}\\
&\qquad+|\alpha_2(t)|\frac{T^{q_1}}{\Gamma(q_1+1)}+|\alpha_4(t)|\Big(|r_1|\frac{\xi^{q_1+1}}{\Gamma(q_1+2)}+|r_2|\frac{T\xi^{q_1}}{\Gamma(q_1+1)}\Big)\Big]\Big\}\\
&+L_2\max_{t\in[0,T]}\frac{1}{|\Delta|}\Big\{|\alpha_2(t)|\Big(|p_1|\frac{\xi^{q_2+1}}{\Gamma(q_2+2)}+|p_2|\frac{T\xi^{q_2}}{\Gamma(q_2+1)}\Big)+|\alpha_3(t)|\frac{T^{q_2-1}}{\Gamma(q_2)}+|\alpha_4(t)|\frac{T^{q_2}}{\Gamma(q_2+1)}\Big\}\\
&\le L_1\mathcal{Q}_1+L_2\mathcal{Q}_2. \qquad (3.4)
\end{aligned}
$$

In a similar fashion, one can obtain that

$$\|\mathcal{H}_2(\varphi,\psi)\|\le L_1\mathcal{Q}_4+L_2\mathcal{Q}_3. \qquad (3.5)$$

From (3.4) and (3.5), it follows that

$$\|\mathcal{H}(\varphi,\psi)\|=\|\mathcal{H}_1(\varphi,\psi)\|+\|\mathcal{H}_2(\varphi,\psi)\|\le L_1(\mathcal{Q}_1+\mathcal{Q}_4)+L_2(\mathcal{Q}_2+\mathcal{Q}_3),$$

which shows that $\mathcal{H}(\Upsilon)$ is uniformly bounded.

To show that $\mathcal{H}(\Upsilon)$ is equicontinuous, let $t_1,t_2\in\mathcal{J}$ with $t_1<t_2$. Then we have

$$
\begin{aligned}
&\big|\mathcal{H}_1(\varphi,\psi)(t_2)-\mathcal{H}_1(\varphi,\psi)(t_1)\big|\\
&\le\Big|\int_0^{t_2}\frac{(t_2-\upsilon)^{q_1-1}}{\Gamma(q_1)}\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon-\int_0^{t_1}\frac{(t_1-\upsilon)^{q_1-1}}{\Gamma(q_1)}\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\Big|\\
&+\frac{|t_2-t_1|}{|\Delta|}\Big\{\Big|\big[2TA_1B_2-T^4-A_2B_2\big]+\big[T^3-2TA_1B_1+A_2B_1\big](t_2+t_1)\Big|\int_0^{T}\frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}\big|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon\\
&\qquad+\big|T^2(t_2+t_1)-2T^3\big|\Big(|p_1|\int_0^{\xi}\frac{(\xi-\upsilon)^{q_2}}{\Gamma(q_2+1)}\big|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon\\
&\qquad+T|p_2|\int_0^{\xi}\frac{(\xi-\upsilon)^{q_2-1}}{\Gamma(q_2)}\big|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon+\int_0^{T}\frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}\big|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon\Big)\\
&\qquad+\Big|\big[2T^2A_2-2T^3A_1\big]+\big[T^2A_1-TA_2\big](t_2+t_1)\Big|\int_0^{T}\frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}\big|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon\\
&\qquad+\Big|2TA_2-4T^2A_1+\big[2TA_1-A_2\big](t_2+t_1)\Big|\Big(|r_1|\int_0^{\xi}\frac{(\xi-\upsilon)^{q_1}}{\Gamma(q_1+1)}\big|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon\\
&\qquad+T|r_2|\int_0^{\xi}\frac{(\xi-\upsilon)^{q_1-1}}{\Gamma(q_1)}\big|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon+\int_0^{T}\frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}\big|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\big|\,d\upsilon\Big)\Big\}\\
&\le\frac{L_1}{\Gamma(q_1+1)}\Big[\big|(t_2-t_1)^{q_1}+t_2^{q_1}-t_1^{q_1}\big|+|(t_2-t_1)^{q_1}|\Big]\\
&+\frac{L_1|t_2-t_1|}{|\Delta|\Gamma(q_1+2)}\Big\{\big|2TA_1B_2-T^4-A_2B_2+\big[T^3-2TA_1B_1+A_2B_1\big](t_2+t_1)\big|q_1(q_1+1)T^{q_1-1}\\
&+\big|T^2(t_2+t_1)-2T^3\big|(q_1+1)T^{q_1}+\big|2TA_2-4T^2A_1+\big[2TA_1-A_2\big](t_2+t_1)\big|\big(|r_1|\xi^{q_1+1}+|r_2|(q_1+1)T\xi^{q_1}\big)\Big\}
\end{aligned}
$$

$$+\frac{L_2|t_2-t_1|}{|\Delta|\Gamma(q_2+2)}\Big\{\big|T^2(t_2+t_1)-2T^3\big|\big(|p_1|\xi^{q_2+1}+|p_2|(q_2+1)T\xi^{q_2}\big)$$
$$+\big|2T^2A_2-2T^3A_1+\big[T^2A_1-TA_2\big](t_2+t_1)\big|q_2(q_2+1)T^{q_2-1}$$
$$+\big|2TA_2-4T^2A_1+\big[2TA_1-A_2\big](t_2+t_1)\big|(q_2+1)T^{q_2}\Big\}\longrightarrow 0$$
$$\text{as } t_2-t_1\to 0 \text{ independently of } (\varphi,\psi)\in\Upsilon.$$

In a similar manner, one can obtain

$$\begin{aligned}
&\big|\mathcal{H}_2(\varphi,\psi)(t_2)-\mathcal{H}_2(\varphi,\psi)(t_1)\big|\\
&\le\Big|\int_0^{t_2}\frac{(t_2-\upsilon)^{q_2-1}}{\Gamma(q_2)}\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon-\int_0^{t_1}\frac{(t_1-\upsilon)^{q_2-1}}{\Gamma(q_2)}\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))\,d\upsilon\Big|\\
&+\frac{|t_2-t_1|}{|\Delta|}\Bigg\{\big|2T^2B_2-2T^3B_1+[T^2B_1-TB_2](t_2+t_1)\big|\int_0^T\frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1+1)}|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))|\,d\upsilon\\
&+\big|2TB_2-4T^2B_1+[2TB_1-B_2](t_2+t_1)\big|\Bigg(|p_1|\int_0^\xi\frac{(\xi-\upsilon)^{q_2}}{\Gamma(q_2+1)}|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))|\,d\upsilon\\
&+T|p_2|\int_0^\xi\frac{(\xi-\upsilon)^{q_2-1}}{\Gamma(q_2)}|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))|\,d\upsilon+\int_0^T\frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))|\,d\upsilon\Bigg)\\
&+\Big|2TA_2B_1-T^4-A_2B_2+[T^3+A_1B_2-2TA_1B_1](t_2+t_1)\Big|\int_0^T\frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))|\,d\upsilon\\
&+\big|T^2(t_2+t_1)-2T^3\big|\Bigg(|r_1|\int_0^\xi\frac{(\xi-\upsilon)^{q_1}}{\Gamma(q_1+1)}|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))|\,d\upsilon\\
&+T|r_2|\int_0^\xi\frac{(\xi-\upsilon)^{q_1-1}}{\Gamma(q_1)}|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))|\,d\upsilon+\int_0^T\frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))|\,d\upsilon\Bigg)\Bigg\}\\
&\le\frac{L_2}{\Gamma(q_2+1)}\Big[\big|(t_2-t_1)^{q_2}+t_2^{q_2}-t_1^{q_2}\big|+|(t_2-t_1)^{q_2}|\Big]\\
&\quad+\frac{L_1|t_2-t_1|}{|\Delta|\Gamma(q_1+2)}\Big\{\big|2T^2B_2-2T^3B_1+[T^2B_1-TB_2](t_2+t_1)\big|q_1(q_1+1)T^{q_1-1}\\
&+\Big|2TB_2-4T^2B_1+[2TB_1-B_2](t_2+t_1)\Big|(q_1+1)T^{q_1}\\
&\quad+\big|T^2(t_2+t_1)-2T^3\big|\big(|r_1|\xi^{q_1+1}+|r_2|(q_1+1)T\xi^{q_1}\big)\Big\}\\
&+\frac{L_2|t_2-t_1|}{|\Delta|\Gamma(q_2+2)}\Big\{\big|2TB_2-4T^2B_1+[2TB_1-B_2](t_2+t_1)\big|\big(|p_1|\xi^{q_2+1}+|p_2|(q_2+1)T\xi^{q_2}\big)\\
&\quad\times\Big|2TA_2B_1-T^4-A_2B_2+\big[T^3+A_1B_2-2TA_1B_1\big](t_2+t_1)\Big|q_2(q_2+1)T^{q_2-1}\\
&+\big|T^2(t_2+t_1)-2T^3\big|(q_2+1)T^{q_2}\Big\}\longrightarrow 0 \text{ as } t_2-t_1\to 0 \text{ independently of } (\varphi,\psi)\in\Upsilon.
\end{aligned}$$

Thus, $\mathcal{H}_1(\Upsilon)$ and $\mathcal{H}_2(\Upsilon)$ are equicontinuous and hence $\mathcal{H}(\Upsilon)$ is equicontinuous. Therefore, we deduce by the Arzelá–Ascoli theorem that $\mathcal{H}(\Upsilon)$ is completely continuous.

In the final step, we consider a set $\Xi=\{(\varphi,\psi)\in\Theta\times\Theta:\ (\varphi,\psi)=\zeta\mathcal{H}(\varphi,\psi),\ 0<\zeta<1\}$ and show that it is bounded. Let $(\varphi,\psi)\in\Xi$. Then $(\varphi,\psi)=\zeta\mathcal{H}(\varphi,\psi)$ implies that $\varphi(t)=\lambda\mathcal{H}_1(\varphi,\psi)(t)$

and $\psi(t) = \zeta\mathcal{H}_2(\varphi,\psi)(t)$ for $t \in \mathcal{J}$. Thus, by the assumption (H_1), we obtain

$$
\begin{aligned}
|\varphi(t)| \le \|\mathcal{H}_1(\varphi,\psi)\| \le \max_{t\in\mathcal{J}} \Bigg\{ \int_0^t \frac{(t-\upsilon)^{q_1-1}}{\Gamma(q_1)} \big[m_0 + m_1|\varphi| + m_2|\psi|\,\big]\, d\upsilon \\
+ \frac{1}{|\Delta|}\Bigg[|\alpha_1(t)| \int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)} \big[m_0 + m_1|\varphi| + m_2|\psi|\,\big]\, d\upsilon \\
+ |\alpha_2(t)|\Bigg(|p_1| \int_0^\xi \frac{(\xi-\upsilon)^{q_2}}{\Gamma(q_2+1)} \big[n_0 + n_1|\varphi| + n_2|\psi|\,\big]\, d\upsilon \\
+ T|p_2| \int_0^\xi \frac{(\xi-\upsilon)^{q_2-1}}{\Gamma(q_2)} \big[n_0 + n_1|\varphi| + n_2|\psi|\,\big]\, d\upsilon + \int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)} \big[m_0 + m_1|\varphi| + m_2|\psi|\,\big]\, d\upsilon\Bigg) \\
+ |\alpha_3(t)| \int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)} \big[n_0 + n_1|\varphi| + n_2|\psi|\,\big]\, d\upsilon \\
+ |\alpha_4(t)|\Bigg(|r_1| \int_0^\xi \frac{(\xi-\upsilon)^{q_1}}{\Gamma(q_1+1)} \big[m_0 + m_1|\varphi| + m_2|\psi|\,\big]\, d\upsilon \\
+ T|r_2| \int_0^\xi \frac{(\xi-\upsilon)^{q_1-1}}{\Gamma(q_1)} \big[m_0 + m_1|\varphi| + m_2|\psi|\,\big]\, d\upsilon + \int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)} \big[n_0 + n_1|\varphi| + n_2|\psi|\,\big]\, d\upsilon\Bigg)\Bigg]\Bigg\},
\end{aligned}
$$

which implies that

$$\|\varphi\| \le \big[m_0 + m_1\|\varphi\| + m_2\|\psi\|\,\big]\mathcal{Q}_1 + \big[n_0 + n_1\|\varphi\| + n_2\|\psi\|\,\big]\mathcal{Q}_2. \tag{3.6}$$

In a similar manner, one can find that

$$\|\psi\| \le \big[n_0 + n_1\|\varphi\| + n_2\|\psi\|\,\big]\mathcal{Q}_3 + \big[m_0 + m_1\|\varphi\| + m_2\|\psi\|\,\big]\mathcal{Q}_4. \tag{3.7}$$

From (3.6), (3.7), it follows that

$$\|\varphi\| + \|\psi\| \le \frac{m_0(\mathcal{Q}_1 + \mathcal{Q}_4) + n_0(\mathcal{Q}_2 + \mathcal{Q}_3)}{\mathcal{Q}_0},$$

where $\mathcal{Q}_0$ is given by (3.3). As a consequence, we have

$$\|(\varphi,\psi)\| \le \frac{m_0(\mathcal{Q}_1 + \mathcal{Q}_4) + n_0(\mathcal{Q}_2 + \mathcal{Q}_3)}{\mathcal{Q}_0}.$$

Therefore, the set Ξ is bounded. Since the hypotheses of the Leray–Schauder nonlinear alternative [17] are satisfied, we deduce by its conclusion that there exists at least one fixed point for the operator $\mathcal{H}$. Hence, problem (1.1), (1.2) admits a solution on $\mathcal{J}$. □

In the following result, we prove the uniqueness of solutions for problem (1.1), (1.2) by applying the Banach fixed point theorem.

Theorem 3.2. *Let the condition* (H_2) *be satisfied. If*

$$(\ell_1 + \ell_2)(\mathcal{Q}_1 + \mathcal{Q}_4) + (\bar{\ell}_1 + \bar{\ell}_2)(\mathcal{Q}_2 + \mathcal{Q}_3) < 1, \tag{3.8}$$

where $\mathcal{Q}_i$, $i = 1,2,3,4$, *are given in* (3.2), *then problem* (1.1), (1.2) *has a unique solution on* $\mathcal{J}$.

Proof. Fixing $\sup_{t\in\mathcal{J}} \rho_1(t,0,0) = N_1 < \infty$ and $\sup_{t\in\mathcal{J}} \rho_2(t,0,0) = N_2 < \infty$, it follows by (H_2) that

$$\begin{aligned} |\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon))| &= \big|\rho_1(\upsilon,\varphi(\upsilon),\psi(\upsilon)) - \rho_1(t,0,0) + \rho_1(t,0,0)\big| \le \ell_1\|\varphi\| + \ell_2\|\psi\| + N_1,\\ |\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon))| &= \big|\rho_2(\upsilon,\varphi(\upsilon),\psi(\upsilon)) - \rho_2(t,0,0) + \rho_2(t,0,0)\big| \le \bar{\ell}_1\|\varphi\| + \bar{\ell}_2\|\psi\| + N_2, \end{aligned} \tag{3.9}$$

Next, we consider a closed ball $\mathcal{U}_\delta = \{(\varphi,\psi)\in\Theta\times\Theta : \|(\varphi,\psi)\| \le \delta\}$ with

$$\delta \ge \frac{N_1(\mathcal{Q}_1+\mathcal{Q}_4) + N_2(\mathcal{Q}_2+\mathcal{Q}_3)}{1 - [(\ell_1+\ell_2)(\mathcal{Q}_1+\mathcal{Q}_4) + (\bar{\ell}_1+\bar{\ell}_2)(\mathcal{Q}_2+\mathcal{Q}_3)]},$$

and show that $\mathcal{H}(\mathcal{U}_\delta) \subset \mathcal{U}_\delta$. For $(\varphi,\psi)\in\Theta\times\Theta$, by using (3.9), we obtain

$$\begin{aligned} \|\mathcal{H}_1(\varphi,\psi)\| \le \max_{t\in\mathcal{J}} \Bigg\{ &\int_0^t \frac{(t-\upsilon)^{q_1-1}}{\Gamma(q_1)} \big[\ell_1\|\varphi\| + \ell_2\|\psi\| + N_1\big]\, d\upsilon \\ &+ \frac{1}{|\Delta|}\Bigg[|\alpha_1(t)| \int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)} \big[\ell_1\|\varphi\| + \ell_2\|\psi\| + N_1\big]\, d\upsilon \\ &+ |\alpha_2(t)|\Bigg(|p_1| \int_0^\xi \frac{(\xi-\upsilon)^{q_2}}{\Gamma(q_2+1)} \big[\bar{\ell}_1\|\varphi\| + \bar{\ell}_2\|\psi\| + N_2\big]\, d\upsilon + T|p_2| \int_0^\xi \frac{(\xi-\upsilon)^{q_2-1}}{\Gamma(q_2)} \big[\bar{\ell}_1\|\varphi\| + \bar{\ell}_2\|\psi\| + N_2\big]\, d\upsilon \\ &+ \int_0^T \frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)} \big[\ell_1\|\varphi\| + \ell_2\|\psi\| + N_1\big]\, d\upsilon\Bigg) + |\alpha_3(t)| \int_0^T \frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)} \big[\bar{\ell}_1\|\varphi\| + \bar{\ell}_2\|\psi\| + N_2\big]\, d\upsilon \\ &+ |\alpha_4(t)|\Bigg(|r_1| \int_0^\xi \frac{(\xi-\upsilon)^{q_1}}{\Gamma(q_1+1)} \big[\ell_1\|\varphi\| + \ell_2\|\psi\| + N_1\big]\, d\upsilon \\ &+ T|r_2| \int_0^\xi \frac{(\xi-\upsilon)^{q_1-1}}{\Gamma(q_1)} \big[\ell_1\|\varphi\| + \ell_2\|\psi\| + N_1\big]\, d\upsilon + \int_0^T \frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)} \big[\bar{\ell}_1\|\varphi\| + \bar{\ell}_2\|\psi\| + N_2\big]\, d\upsilon\Bigg)\Bigg]\Bigg\} \\ &\le \big[(\ell_1+\ell_2)\delta + N_1\big]\mathcal{Q}_1 + \big[(\bar{\ell}_1+\bar{\ell}_2)\delta + N_2\big]\mathcal{Q}_2. \end{aligned}$$

Likewise, we have

$$\|\mathcal{H}_2(\varphi,\psi)\| \le \big[(\bar{\ell}_1+\bar{\ell}_2)\delta + N_2\big]\mathcal{Q}_3 + \big[(\ell_1+\ell_2)\delta + N_1\big]\mathcal{Q}_4.$$

Therefore, we get

$$\begin{aligned} \|\mathcal{H}(\varphi,\psi)\| &= \|\mathcal{H}_1(\varphi,\psi)\| + \|\mathcal{H}_2(\varphi,\psi)\| \\ &\le \Big[(\ell_1+\ell_2)(\mathcal{Q}_1+\mathcal{Q}_4) + (\bar{\ell}_1+\bar{\ell}_2)(\mathcal{Q}_2+\mathcal{Q}_3)\Big]\delta + (\mathcal{Q}_1+\mathcal{Q}_4)N_1 + (\mathcal{Q}_2+\mathcal{Q}_3)N_2 \le \delta, \end{aligned}$$

which shows that $\mathcal{H}(\varphi,\psi) \in \mathcal{U}_\delta$. Hence $\mathcal{H}(\mathcal{U}_\delta) \subset \mathcal{U}_\delta$.

Now, we will establish that the operator $\mathcal{H}$ is a contraction. Toward this end, let $(\varphi_1,\psi_1),(\varphi_2,\psi_2) \in \Theta\times\Theta$. Then, for any $t\in\mathcal{J}$, we obtain

$$\begin{aligned} \big\|\mathcal{H}_1(\varphi_2,\psi_2) - \mathcal{H}_1(\varphi_1,\psi_1)\big\| \le \max_{t\in\mathcal{J}} \Bigg\{ &\int_0^t \frac{(t-\upsilon)^{q_1-1}}{\Gamma(q_1)} \big|\rho_1(\upsilon,\varphi_2(\upsilon),\psi_2(\upsilon)) - \rho_1(\upsilon,\varphi_1(\upsilon),\psi_1(\upsilon))\big|\, d\upsilon \\ &+ \frac{1}{|\Delta|}\Bigg[|\alpha_1(t)| \int_0^T \frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)} \big|\rho_1(\upsilon,\varphi_2(\upsilon),\psi_2(\upsilon)) - \rho_1(\upsilon,\varphi_1(\upsilon),\psi_1(\upsilon))\big|\, d\upsilon \end{aligned}$$

$$+|\alpha_2(t)|\Big(|p_1|\int_0^{\xi}\frac{(\xi-\upsilon)^{q_2}}{\Gamma(q_2+1)}\big|\rho_2(\upsilon,\varphi_2(\upsilon),\psi_2(\upsilon))-\rho_2(\upsilon,\varphi_1(\upsilon),\psi_1(\upsilon))\big|\,d\upsilon$$

$$+T|p_2|\int_0^{\xi}\frac{(\xi-\upsilon)^{q_2-1}}{\Gamma(q_2)}\big|\rho_2(\upsilon,\varphi_2(\upsilon),\psi_2(\upsilon))-\rho_2(\upsilon,\varphi_1(\upsilon),\psi_1(\upsilon))\big|\,d\upsilon$$

$$+\int_0^{T}\frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}\big|\rho_1(\upsilon,\varphi_2(\upsilon),\psi_2(\upsilon))-\rho_1(\upsilon,\varphi_1(\upsilon),\psi_1(\upsilon))\big|\,d\upsilon\Big)$$

$$+|\alpha_3(t)|\int_0^{T}\frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}\big|\rho_2(\upsilon,\varphi_2(\upsilon),\psi_2(\upsilon))-\rho_2(\upsilon,\varphi_1(\upsilon),\psi_1(\upsilon))\big|\,d\upsilon$$

$$+|\alpha_4(t)|\Big(|r_1|\int_0^{\xi}\frac{(\xi-\upsilon)^{q_1}}{\Gamma(q_1+1)}\big|\rho_1(\upsilon,\varphi_2(\upsilon),\psi_2(\upsilon))-\rho_1(\upsilon,\varphi_1(\upsilon),\psi_1(\upsilon))\big|\,d\upsilon$$

$$+T|r_2|\int_0^{\xi}\frac{(\xi-\upsilon)^{q_1-1}}{\Gamma(q_1)}\big|\rho_1(\upsilon,\varphi_2(\upsilon),\psi_2(\upsilon))-\rho_1(\upsilon,\varphi_1(\upsilon),\psi_1(\upsilon))\big|\,d\upsilon$$

$$+\int_0^{T}\frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}\big|\rho_2(\upsilon,\varphi_2(\upsilon),\psi_2(\upsilon))-\rho_2(\upsilon,\varphi_1(\upsilon),\psi_1(\upsilon))\big|\,d\upsilon\Big)\Big]\Big\}$$

$$\leq\big(\ell_1\|\varphi_2-\varphi_1\|+\ell_2\|\psi_2-\psi_1\|\big)\max_{t\in\mathcal{J}}\Big\{\int_0^{t}\frac{(t-\upsilon)^{q_1-1}}{\Gamma(q_1)}\,d\upsilon+\frac{1}{|\Delta|}\Big[|\alpha_1(t)|\int_0^{T}\frac{(T-\upsilon)^{q_1-2}}{\Gamma(q_1-1)}\,d\upsilon$$

$$+|\alpha_2(t)|\Big(|p_1|\int_0^{\xi}\frac{(\xi-\upsilon)^{q_2}}{\Gamma(q_2+1)}\,d\upsilon+T|p_2|\int_0^{\xi}\frac{(\xi-\upsilon)^{q_2-1}}{\Gamma(q_2)}\,d\upsilon+\int_0^{T}\frac{(T-\upsilon)^{q_1-1}}{\Gamma(q_1)}\,d\upsilon\Big)$$

$$+|\alpha_3(t)|\int_0^{T}\frac{(T-\upsilon)^{q_2-2}}{\Gamma(q_2-1)}\,d\upsilon$$

$$+|\alpha_4(t)|\Big(|r_1|\int_0^{\xi}\frac{(\xi-\upsilon)^{q_1}}{\Gamma(q_1+1)}\,d\upsilon+T|r_2|\int_0^{\xi}\frac{(\xi-\upsilon)^{q_1-1}}{\Gamma(q_1)}\,d\upsilon+\int_0^{T}\frac{(T-\upsilon)^{q_2-1}}{\Gamma(q_2)}\,d\upsilon\Big)\Big]\Big\}$$

$$\leq\big[(\ell_1+\ell_2)\mathcal{Q}_1+(\bar{\ell}_1+\bar{\ell}_2)\mathcal{Q}_2\big]\big(\|\varphi_2-\varphi_1\|+\|\psi_2-\psi_1\|\big).$$

Similarly, we can get

$$\big\|\mathcal{H}_2(\varphi_2,\psi_2)-\mathcal{H}_2(\varphi_1,\psi_1)\big\|\leq\big[(\ell_1+\ell_2)\mathcal{Q}_4+(\bar{\ell}_1+\bar{\ell}_2)\mathcal{Q}_3\big]\big(\|\varphi_2-\varphi_1\|+\|\psi_2-\psi_1\|\big).$$

From the last two inequalities, it follows that

$$\big\|\mathcal{H}(\varphi_2,\psi_2)-\mathcal{H}(\varphi_1,\psi_1)\big\|\leq\big[(\ell_1+\ell_2)(\mathcal{Q}_1+\mathcal{Q}_4)+(\bar{\ell}_1+\bar{\ell}_2)(\mathcal{Q}_2+\mathcal{Q}_3)\big]\big(\|\varphi_2-\varphi_1\|+\|\psi_2-\psi_1\|\big),$$

which, in view of condition (3.8), implies that the operator $\mathcal{H}$ is a contraction. So, by the Banach fixed point theorem, there exists a unique fixed point for the operator $\mathcal{H}$ which is indeed a unique solution to problem (1.1), (1.2) on $\mathcal{J}$. □

3.1 Example

Consider a fully coupled fractional boundary value problem:

$$\begin{cases} {}^{C}D^{2.03}\varphi(t) = \rho_1(t,\varphi(t),\psi(t)), \quad t\in[0,2], \\ {}^{C}D^{2.51}\psi(t) = \rho_2(t,\varphi(t),\psi(t)), \quad t\in[0,2] \\ \varphi(0)=0, \quad \varphi'(2)=0, \quad \varphi(2) = \int\limits_0^{\frac{3}{2}} \Big[\frac{5}{2}\,\psi(s) + \frac{8}{5}\,\psi'(s)\Big]\,ds, \\ \psi(0)=0, \quad \psi'(2)=0, \quad \psi(2) = \int\limits_0^{\frac{3}{2}} \Big[\frac{3}{10}\,\varphi(s) + \frac{4}{5}\,\varphi'(s)\Big]\,ds, \end{cases} \tag{3.10}$$

where $T=2$, $q_1=2.03$, $q_2=2.51$, $p_1=\frac{5}{2}$, $p_2=\frac{4}{5}$, $r_1=\frac{3}{10}$, $r_2=\frac{2}{5}$. With the given data, it is found that $\mathcal{Q}_1\approx 1.448$, $\mathcal{Q}_2\approx 3.6000$, $\mathcal{Q}_3\approx 1.332$ and $\mathcal{Q}_4\approx 1.141$ ($\mathcal{Q}_i$, $i=1,2,3,4$, are given in (3.2)).

(a) We illustrate Theorem 3.1 by choosing

$$\begin{aligned} \rho_1(t,\varphi(t),\psi(t)) &= \frac{1}{2(t^2+6)}\,\frac{\varphi(t)|\varphi(t)|}{(1+|\varphi(t)|)} + \frac{1}{15}\,\psi(t)\cos\psi(t) + \frac{1}{9\sqrt{t^2+9}}\,, \\ \rho_2(t,\varphi(t),\psi(t)) &= \frac{1}{23}\,\frac{\varphi(t)|\psi(t)|}{(1+|\psi(t)|)} + \frac{\psi(t)\cos\varphi(t)}{(7+t^2)^2} + \frac{1}{3(t+4)^2}\,. \end{aligned} \tag{3.11}$$

From (3.11), it is easy to find that $m_0=\frac{1}{27}$, $m_1=\frac{1}{12}$, $m_2=\frac{1}{15}$, $n_0=\frac{1}{48}$, $n_1=\frac{1}{23}$, $n_2=\frac{1}{49}$. Moreover,

$$m_1(\mathcal{Q}_1+\mathcal{Q}_4)+n_1(\mathcal{Q}_2+\mathcal{Q}_3)\approx 0.430<1 \ \text{ and } \ m_2(\mathcal{Q}_1+\mathcal{Q}_4)+n_2(\mathcal{Q}_2+\mathcal{Q}_3)\approx 0.273<1.$$

Thus the hypotheses of Theorem 3.1 are satisfied and, consequently, there exists at least one solution to problem (3.10) with $\rho_1(t,\varphi(t),\psi(t))$ and $\rho_2(t,\varphi(t),\psi(t))$ given by (3.11) on $[0,2]$.

(b) For demonstrating the application of Theorem 3.2, we take

$$\begin{aligned} \rho_1(t,\varphi(t),\psi(t)) &= \frac{e^{-t^2}}{(t^2+40)}\,\tan^{-1}\varphi(t) + \frac{1}{(35+t^3)}\,\frac{|\psi(t)|}{(1+|\psi(t)|)} + \frac{\cos t}{8\sqrt{t^3+2}}\,, \\ \rho_2(t,\varphi(t),\psi(t)) &= \frac{1}{(t^4+36)}\,\frac{|\varphi(t)|}{(1+|\varphi(t)|)} + \frac{1}{\sqrt{t^2+625}}\,\cos\psi(t) + \frac{e^t}{2(\cos^2 t+7)}\,. \end{aligned} \tag{3.12}$$

It follows from (3.12) that $\ell_1=\frac{1}{40}$, $\ell_2=\frac{1}{35}$, $\overline{\ell}_1=\frac{1}{36}$, $\overline{\ell}_2=\frac{1}{25}$ and

$$\big[(\ell_1+\ell_2)(Q_1+Q_4)+(\overline{\ell}_1+\overline{\ell}_2)(Q_2+Q_3)\big]\approx 0.473<1.$$

Clearly, the hypothesis of Theorem 3.2 is satisfied and hence its conclusion implies that there exists a unique solution to problem (3.10) with $\rho_1(t,\varphi(t),\psi(t))$ and $\rho_2(t,\varphi(t),\psi(t))$ given in (3.12) on $[0,2]$.

4 Conclusion

In this paper, we have obtained the existence and uniqueness results for a Caputo type nonlinear fractional differential system supplemented with a new class of integral boundary conditions. The standard fixed point theorems are the main tools of our study. As a special case, our results correspond to the integral boundary conditions of the form

$$\varphi(0)=0,\ \ \varphi'(T)=0,\ \ \varphi(T)=\int\limits_0^{T^-}\big[p_1\psi(s)+Tp_2\psi'(s)\big]\,ds = p_1\int\limits_0^{T^-}\psi(s)\,ds+Tp_2\psi(T),$$

$$\psi(0)=0,\ \ \psi'(T)=0,\ \ \psi(T)=\int_0^{T^-}\big[r_1\varphi(s)+Tr_2\varphi'(s)\big]\,ds=r_1\int_0^{T^-}\varphi(s)\,ds+Tr_2\varphi(T),$$

in the limit $\xi\to T^-$, which are indeed new. If we take $p_1=0=r_1$ in the results of this paper, then we obtain the new ones associated with the following boundary conditions:

$$\varphi(0)=0,\ \ \varphi'(T)=0,\ \ \varphi(T)=Tp_2\int_0^{\xi}\psi'(s)\,ds=Tp_2\psi(\xi),$$

$$\psi(0)=0,\ \ \psi'(T)=0,\ \ \psi(T)=Tr_2\int_0^{\xi}\varphi'(s)\,ds=Tr_2\varphi(\xi).$$

Thus, our results are not only new in the given configuration but also give rise to some new results as special cases by fixing the parameters involved in the problem at hand.

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References

[1] R. Agarwal, S. Hristova and D. O'Regan, Integral presentations of the solution of a boundary value problem for impulsive fractional integro-differential equations with Riemann-Liouville derivatives. *AIMS Math.* **7** (2022), no. 2, 2973–2988.

[2] B. Ahmad, M. Alnahdi, S. K. Ntouyas and A. Alsaedi, On a mixed nonlinear fractional boundary value problem with a new class of closed integral boundary conditions. *Qual. Theory Dyn. Syst.* **22** (2023), no. 3, Paper no. 96, 17 pp.

[3] B. Ahmad, A. Alsaedi, Y. Alruwaily and S. K. Ntouyas, Nonlinear multi-term fractional differential equations with Riemann-Stieltjes integro-multipoint boundary conditions. *AIMS Math.* **5** (2020), no. 2, 1446–1461.

[4] B. Ahmad, J. Henderson and R. Luca, *Boundary Value Problems for Fractional Differential Equations and Systems.* Trends in Abstract and Applied Analysis, 9. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2021.

[5] B. Ahmad and S. K. Ntouyas, *Nonlocal Nonlinear Fractional-Order Boundary Value Problems.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2021.

[6] A. Alsaedi, M. Alnahdi, B. Ahmad and S. K. Ntouyas, On a nonlinear coupled Caputo-type fractional differential system with coupled closed boundary conditions. *AIMS Math.* **8** (2023), no. 8, 17981–17995.

[7] D. ben-Avraham and S. Havlin, *Diffusion and Reactions in Fractals and Disordered Systems.* Cambridge University Press, Cambridge, 2000.

[8] J. R. Cannon, The solution of the heat equation subject to the specification of energy. *Quart. Appl. Math.* **21** (1963), 155–160.

[9] A. Carvalho and C. M. A. Pinto, A delay fractional order model for the co-infection of malaria and HIV/AIDS. *Int. J. Dyn. Control* **5** (2017), no. 1, 168–186.

[10] R. Yu. Chegis, Numerical solution of a heat conduction problem with an integral condition. (Russian) *Litovsk. Mat. Sb.* **24** (1984), no. 4, 209–215.

[11] R. Čiegis and A. Bugajev, Numerical approximation of one model of bacterial self-organization. *Nonlinear Anal. Model. Control* **17** (2012), no. 3, 253–270.

[12] A. S. Deshpande and V. Daftardar-Gejji, On disappearance of chaos in fractional systems. *Chaos Solitons Fractals* **102** (2017), 119–126.

[13] Y. Ding, Z. Wang and H. Ye, Optimal control of a fractional-order HIV-immune system with memory. *IEEE Transactions on Control Systems Technology* **20** (2012), no. 3, 763–769.

[14] T. S. Doan and P. E. Kloeden, Semi-dynamical systems generated by autonomous Caputo fractional differential equations. *Vietnam J. Math.* **49** (2021), no. 4, 1305–1315.

[15] M. Faieghi, S. Kuntanapreeda, H. Delavari and D. Baleanu, LMI-based stabilization of a class of fractional-order chaotic systems. *Nonlinear Dynam.* **72** (2013), no. 1-2, 301–309.

[16] H. A. Fallahgoul, S. M. Focardi and F. J. Fabozzi, *Fractional Calculus and Fractional Processes with Applications to Financial Economics. Theory and Application.* Elsevier/Academic Press, London, 2017.

[17] A. Granas and J. Dugundji, *Fixed Point Theory.* Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[18] C. Guendouz, J. E. Lazreg, J. J. Nieto and A. Ouahab, Existence and compactness results for a system of fractional differential equations. *J. Funct. Spaces* **2020**, Art. ID 5735140, 12 pp.

[19] D. He and L. Xu, Exponential stability of impulsive fractional switched systems with time delays. *IEEE Transactions on Circuits and Systems II: Express Briefs* **68** (2021), no. 6, 1972–1976.

[20] J. Henderson, Smoothness of solutions with respect to multi-strip integral boundary conditions for nth order ordinary differential equations. *Nonlinear Anal. Model. Control* **19** (2014), no. 3, 396–412.

[21] S. Hristova, R. Agarwal and D. O'Regan, Explicit solutions of initial value problems for systems of linear Riemann-Liouville fractional differential equations with constant delay. *Adv. Difference Equ.* **2020**, Paper no. 180, 18 pp.

[22] N. I. Ionkin, The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition. (Russian) *Differencial'nye Uravnenija* **13** (1977), no. 2, 294–304.

[23] G. Iskenderoglu and D. Kaya, Symmetry analysis of initial and boundary value problems for fractional differential equations in Caputo sense. *Chaos Solitons Fractals* **134** (2020), 109684, 7 pp.

[24] M. Javidi and B. Ahmad, Dynamic analysis of time fractional order phytoplankton–toxic phytoplankton–zooplankton system. *Ecological Modelling* **318** (2015), 8–18.

[25] P. Kang, Positive solutions for a singular system of nonlinear fractional differential equations. *Stud. Sci. Math. Hung.* **59** (2022), no. 3-4, 183–195.

[26] C. Kiataramkul, S. K. Ntouyas and J. Tariboon, Existence results for ψ-Hilfer fractional integro-differential hybrid boundary value problems for differential equations and inclusions. *Adv. Math. Phys.* **2021**, Art. ID 9044313, 12 pp.

[27] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[28] K. D. Kucche and A. D. Mali, On the nonlinear (k, Ψ)-Hilfer fractional differential equations. *Chaos Solitons Fractals* **152** (2021), Paper no. 111335, 14 pp.

[29] Z. Laadjal and F. Jarad, Existence, uniqueness and stability of solutions for generalized proportional fractional hybrid integro-differential equations with Dirichlet boundary conditions. *AIMS Math.* **8** (2023), no. 1, 1172–1194.

[30] R. Luca, On a system of fractional differential equations with p-Laplacian operators and integral boundary conditions. *Rev. Roumaine Math. Pures Appl.* **66** (2021), no. 3-4, 749–766.

[31] L. Ma and B. Wu, On the fractional Lyapunov exponent for Hadamard-type fractional differential system. *Chaos* **33** (2023), no. 1, Paper no. 013117, 9 pp.

[32] F. Nicoud and T. Schönfeld, Integral boundary conditions for unsteady biomedical CFD applications. *Int. J. Numer. Methods Fluids* **40** (2002), no. 3-4, 457–465.

[33] J. J. Nieto, Fractional Euler numbers and generalized proportional fractional logistic differential equation. *Fract. Calc. Appl. Anal.* **25** (2022), no. 3, 876–886.

[34] M. Syed Ali, G. Narayanan, V. Shekher, A. Alsaedi and B. Ahmad, Global Mittag-Leffler stability analysis of impulsive fractional-order complex-valued BAM neural networks with time varying delays. *Commun. Nonlinear Sci. Numer. Simul.* **83** (2020), 105088, 22 pp.

[35] V. V. Tarasova and V. E. Tarasov, Logistic map with memory from economic model. *Chaos Solitons Fractals* **95** (2017), 84–91.

[36] Ch. A. Taylor, T. J. R. Hughes and Ch. K. Zarins, Finite element modeling of blood flow in arteries. *Comput. Methods Appl. Mech. Engrg.* **158** (1998), no. 1-2, 155–196.

[37] S. Wang and M. Xu, Axial Couette flow of two kinds of fractional viscoelastic fluids in an annulus. *Nonlinear Anal. Real World Appl.* **10** (2009), no. 2, 1087–1096.

[38] J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems involving integral conditions. *NoDEA Nonlinear Differential Equations Appl.* **15** (2008), no. 1-2, 45–67.

[39] A. Wongcharoen, S. K. Ntouyas, P. Wongsantisuk and J. Tariboon, Existence results for a nonlocal coupled system of sequential fractional differential equations involving ψ-Hilfer fractional derivatives. *Adv. Math. Phys.* **2021**, Art. ID 5554619, 9 pp.

[40] L. Xu, X. Chu and H. Hu, Exponential ultimate boundedness of non-autonomous fractional differential systems with time delay and impulses. *Appl. Math. Lett.* **99** (2020), 106000, 7 pp.

[41] Y. Xu and W. Li, Finite-time synchronization of fractional-order complex-valued coupled systems. *Phys. A* **549** (2020), 123903, 11 pp.

[42] Y. Xu, Y. Li and W. Li, Adaptive finite-time synchronization control for fractional-order complex-valued dynamical networks with multiple weights. *Commun. Nonlinear Sci. Numer. Simul.* **85** (2020), 105239, 17 pp.

[43] G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics.* Reprint of the 2005 original. Oxford University Press, Oxford, 2008.

[44] F. Zhang, G. Chen, C. Li and J. Kurths, Chaos synchronization in fractional differential systems. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **371** (2013), no. 1990, 20120155, 26 pp.

[45] L. Zheng and X. Zhang, *Modeling and Analysis of Modern Fluid Problems.* Mathematics in Science and Engineering. Elsevier/Academic Press, London, 2017.

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